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GENERALIZED GAUSSIAN QUADRATURE FORMULAS WITH CHEBYSHEV NODES*1)

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Abstract

Explicit expressions of the Cotes numbers of the generalized Gaussian quadrature formulas for the Chebyshev nodes (of the first kind and the second kind) and their asymptotic behavior are given.

Key words: Quadrature formula, Chebyshev polynomials.

1. Introduction

This paper deals with the generalized Gaussian quadrature formulas for Chebyshev nodes (cf. [2]).

Throughout the paper we assume that m and n are positive integers. As usually, $T_n(x)$ and $U_n(x)$ denote the n-th Chebyshev polynomials of the first kind and the second kind, respectively. Among generalized Gaussian quadrature formulas one of the most important cases is the weight

$$w_m(x) := (1-x^2)^{[(m+1)/2]-(m+1)/2}, \tag{1.1}$$

where $[r]$ denotes the largest integer $\le r$. In [5] we pointed out that if we take as nodes of a quadrature formula the zeros of $(1-x^2)U_{n-1}(x)$ (here we replace $n+1$ by n for convenience)

$$x_{kn} = \cos\frac{k\pi}{n}, \quad k = 0, 1, ..., n, \tag{1.2}$$

then the quadrature formula with certain numbers $c_{ikm} := c_{ikmn}$ (called Cotes numbers of higher order)

$$\int_{-1}^{1} f(x)\sigma_m(x)w_m(x)dx = \sum_{k=0}^{n}\sum_{i=0}^{m_k} c_{ikm} f^{(i)}(x_k) \tag{1.3}$$

is exact for all $f \in \boldsymbol{P}_{mn+[(-1)^m-3]/2}$, where

$$\sigma_m(x) := \text{sgn } U_{n-1}(x)^m \tag{1.4}$$

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and

$$m_k := [n_k(m-2)], \quad n_k := \begin{cases} 1, & 1 \le k \le n-1, \\ \frac{1}{2}, & k = 0, n. \end{cases} \tag{1.5}$$

As it turns out, the most interesting property of this quadrature formula is that its nodes do not depend on the index m. In [5] we found the explicit formulas for c_{ikmn} and their asymptotic behavoiur as $n \to \infty$, which provided an answer to an analogue of Problem 26 of Turán [6, p. 47]. To state this results, which will be used later, we put:

$$\Delta_m(x) = (1-x^2)^{[m/2]} U_{n-1}(x)^m, \tag{1.6}$$

$$d_{km} = \Delta_m^{([n_k m])}(x_k) = \begin{cases} m!(1-x_k^2)^{[m/2]} U'_{n-1}(x_k)^m, & 1 \le k \le n-1, \\ (-2)^{[m/2]} \left(\left[\frac{m}{2}\right]\right)! U_{n-1}(1)^m, & k = 0, \\ 2^{[m/2]} \left(\left[\frac{m}{2}\right]\right)! U_{n-1}(-1)^m, & k = n, \end{cases} \tag{1.7}$$

$$L_{km}(x) = \frac{([n_k m])! \Delta_m(x)}{d_{km}(x-x_k)^{[n_k m]}}, \quad k = 0, 1, ..., n, \tag{1.8}$$

$$b_{ikm} = \frac{1}{i!} \left[L_{km}(x)^{-1} \right]^{(i)}_{x=x_k}, \quad i = 0, 1, ...; \quad k = 0, 1, ..., n, \tag{1.9}$$

$$B_{ikm} = \frac{1}{i!} \left\{ \sum_{\nu \in \{0,n\} \setminus \{k\}} [2(x_\nu - x) L_{km}(x)]^{-1} \right\}^{(i)}_{x=x_k}, \quad i = 0, 1, ...; \quad k = 0, 1, ..., n, \tag{1.10}$$

$$s_m = \begin{cases} 2, & \text{if } m \text{ is odd}, \\ \pi, & \text{if } m \text{ is even}. \end{cases} \tag{1.11}$$

Then we have (cf. [5]; for $m = 4$ the results of the theorem can be found in [7]).

Theorem A. *Let* (1.2) *be given. Then for each* $k, 0 \le k \le n$, *and for each* i, $0 \le i \le m_k$,

$$\begin{cases} c_{m_k,k,m} = \dfrac{n_k s_m (m-2)!}{d_{k,m-2}[(m-2)!!]^2 n}, & m \ge 2, \\ c_{m_k+1,k,m} = 0, \end{cases} \tag{1.12}$$

$$c_{ikm} = c_{i,k,m-2} + \frac{m_k! c_{m_k,k,m}}{i! n_k (m-2)} \{ (i + n_k(m-2) - m_k) b_{m_k - i, k, m-2} - \frac{1}{2}[1 + (-1)^{m+1}] B_{m_k - i - 1, k, m-2} \}, \quad m \ge 3. \tag{1.13}$$