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BOUNDARY ELEMENT APPROXIMATION OF STEKLOV EIGENVALUE PROBLEM FOR HELMHOLTZ EQUATION*1)2)

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Abstract

Steklov eigenvalue problem of Helmholtz equation is considered in the present paper. Steklov eigenvalue problem is reduced to a new variational formula on the boundary of a given domain, in which the self-adjoint property of the original differential operator is kept and the calculating of hyper-singular integral is avoided. A numerical example showing the efficiency of this method and an optimal error estimate are given.

Key words: Steklov eigenvalue problem, differential operator, error estimate, boundary element approximation.

1. Introduction

We consider the following Steklov eigenvalue problem:

Find nonzero u and number λ, such that

$$\begin{aligned} &-\Delta u + u = 0, \quad \text{in} \ \ \Omega, \\ &\frac{\partial u}{\partial n} = \lambda u, \quad \text{on} \ \ \Gamma, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded domain with sufficient smooth boundary Γ, $\frac{\partial}{\partial n}$ is the outward normal derivative on Γ.

Courant and Hilbert[1] studied the following eigenvalue problem:

$$\Delta u = 0, \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = \lambda u, \quad \text{on } \Gamma, \tag{1.2}$$

which was reduced to the eigenvalue problem of an integral equation by using the Green's function of $\Delta u = 0$ with Nuemann boundary condition. From Fredholm theorem, we know that (1) the problem (1.2) has infinite number of eigenvalues, which are all real numbers, (2) suppose that $u_n(x)$, $u_m(x)$ are two eigenvalues of the problem (1.2) corresponding two different eigenvalues λ_n and λ_m, then

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$$\int_\Gamma u_n(x)u_m(x)ds_x = 0, \tag{1.3}$$

i.e. the trace of $u_n(x)$ and $u_m(x)$ on Γ are orthogonal on the space of $L^2(\Gamma)$.

Moreover, Courant and Hilbert[1] pointed out that analogous considerations held for the general self-adjoint second order elliptic differential equation, so for the problem (1.1).

But it is difficult to obtain the numerical solution of the problem (1.1), or (1.2) by the integral formula given by Courant and Hilbert. The reason is that for only a few of special domains, the Green's function is known. Bramble and Osborn[2] developed a finite element method for the Steklov eigenvalue problem and the optimal error estimate was given. Han, Guan and He discussed the boundary element approximation of the problem (1.2) [9] and the error estimate was given in [10] by Han and Guan. In this paper, a equivalent variational formula on the boundary Γ for the problem (1.1) is proposed, using the fundamental solution of $-\Delta u + u = 0$. Then the boundary finite element approximation of the problem (1.1) was obtain. A numerical example shows that the new method is very efficient.

2. A New Variational Formula on the Boundary Γ of Problem (1.1) and Its Boundary Element Approximation

The fundamental solution of equation $-\Delta u + u = 0$ in Ω is the modified Bessel function of zero order $K_0(|x-y|)$, which is given by

$$K_0(r) = \frac{\pi i}{2} H_0^{(1)}(ir) = \sum_{n=0}^{\infty} a_n r^{2n} \log\frac{1}{r} + \sum_{n=1}^{\infty} b_n r^{2n},\ a_0 = 1, \tag{2.1}$$

with a_n, b_n $(n = 1, 2, \cdots)$ unique determined nearby $r = 0$ and, we have

$$K_0(r) = \sqrt{\frac{\pi}{2r}} e^{-r} + \cdots, \tag{2.2}$$

at infinity. So $\lim\limits_{r\to+\infty} K_0(r) = 0$. $K_0(r)$ satisfies the following differential equation

$$\frac{d^2K_0(r)}{dr^2} + \frac{1}{r}\frac{dK_0(r)}{dr} - K_0(r) = 0, \quad r \neq 0. \tag{2.3}$$

By using Green's formula it is obtained:

$$u(x) = -\frac{1}{2\pi}\int_\Gamma u(y)\frac{\partial K_0(|x-y|)}{\partial n_y} ds_y + \frac{1}{2\pi}\int_\Gamma p(y) K_0(|x-y|) ds_y, \quad \forall x \in \Omega, \tag{2.4}$$

where $u(x)$ is any solution of equation $-\Delta u + u = 0$, $p(y) = \frac{\partial u(y)}{\partial n_y}\Big|_\Gamma$, and n_y denotes the outward unit normal to Γ at point y. The formula (2.4) shows that every function u satisfying $-\Delta u + u = 0$ in Ω and continuously differentiable on $\Omega + \Gamma$ can be represented as the potential of a distribution on the boundary Γ consisting of a single-layer of density