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# THE THEOREMS ON THE B-B POLYNOMIALS DEFINED ON A SIMPLEX IN THE BLOSSOMING FORM*

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**Abstract**

In this paper, an explicit expression of the blossom by the means of the B–form of B-B polynomial defined on a simplex is given. With its help, new but very short and simple blossom proofs of the most important theorems on B-B polynomials are derived, such as the degree elevation formula, the subdivision and the change of the underlying simplex procedure, the control points convergence property, the Marsden's identity.

## 1. Introduction

There are many approaches to polynomial and piecewise polynomial functions and curves. In the late '80 an elegant new approach has been developed by Ramshaw and others under the name of "blossoming"[7,8,2,3]. The idea applied can be traced to the algebraic geometry under the name *polar form*. As it turned out, the new approach successfully reconstructs and generalises the standard univariate polynomial and spline theory and makes it easier to understand and to explain the theorems and the computational algorithms involved. This is mainly due to the fact that it is possible to derive the main properties of splines (and polynomials) just from the recurrence relation that computes the B-spline basis. And this recurrence is a particular case of an algorithm that computes a blossom.

The basic idea in the blossom approach is the conclusion that there is a one–to–one correspondence between polynomials of degree at most $n$ and a certain class of polynomials of $n$ variables. Let us be precise.

A function $f : \mathbb{R} \to \mathbb{R}$ is called affine if it satisfies

$$f\left(\sum_i u_i x_i\right) = \sum_i u_i f(x_i)$$

for all affine combinations of $x_i \in \mathbb{R}$, i.e.

$$\sum_i u_i = 1, \quad u_i \in \mathbb{R}.$$

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A function $f : \mathbb{R}^m \to \mathbb{R}$ is called $m$–affine if it is affine with respect to each arguments. Also, a function $f : \mathbb{R}^m \to \mathbb{R}$ is called symmetric if it preserves its value under any permutation of its argument. Let us give now the definition of a blossom for the univariate polynomial case.

**Definition 1.1** *Let $f$ be a polynomial of degree $\leq n$. The blossom*

$$B_f(u_1, u_2, \cdots, u_n)$$

*of the polynomial $f$ is the symmetric $n$–affine multivariate polynomial satisfying the diagonal property*

$$B_f(t, t, \cdots, t) = f(t) \ .$$

The blossom is well defined since it turns out to be unique. The definition can be straightforwardly extended to the splines, polynomial and spline curves, as well as to the multivariate polynomial case. In the recent years, many authors studied the problems on the univariate splines and on the spline or more general progressive curves using blossoming approach and produced fertile results[9,10,6,1,11,4]. The results were partly extended also to the multivariate polynomial case. But on the other hand, the analogies of blossoming for general multivariate splines are currently not known and this problem remains an important open question. In [5], some blossoming facts concerning splines on a simplicial partition can be found.

In this paper, we give an explicit expression of blossom

$$B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) := B_n(b_n; x^{(1)}, x^{(2)}, \cdots, x^{(n)})$$

by means of the B–form of B-B polynomial $b_n$ on a simplex. Compared to the known one[5] it turns out to be much simpler to deal with, and gives the important dual functional property in a natural one row proof. It is also very easy to establish a necessary and sufficient condition (in the blossom form) for two B-B polynomials $b_n$ and $b_{n+k}$ to be identical. On this basis, the paper continues with very short blossom proofs of the most important theorems on B-B polynomials defined on simplex.

Although the majority of the theorems in this article is known, the proofs in the outline are new. The theorems are formulated in the blossoming form and the proofs are much shorter than the currently known ones and may give further insight into the basic theory of B-B surfaces and multivariate splines.

## 2. The Blossoming Proofs of the Facts on the B-B Polynomials

In the beginning, let us introduce the notation used throughout the paper. Let

$$V_m := < v^{(1)}, v^{(2)}, \cdots, v^{(m+1)} > := \left\{ \sum_{i=1}^{m+1} \lambda_i v^{(i)} : \lambda_i \geq 0, \quad \sum_{i=1}^{m+1} \lambda_i = 1 \right\}$$

denote a non–degenerate simplex in $\mathbb{R}^m$ with vertices

$$v^{(1)}, v^{(2)}, \cdots, v^{(m+1)} \in \mathbb{R}^m, \ \ \text{volume } V_m \neq 0.$$