Journal of Computational Mathematics, Vol.12, No.2, 1994, 123–131.

$(0,1,\cdots,m-2,m)$ INTERPOLATION FOR THE LAGUERRE ABSCISSAS*[1)]

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Abstract

A necessary and sufficient condition of regularity of $(0,1,\cdots,m-2,m)$ interpolation on the zeros of the Laguerre polynomials $L_n^{(\alpha)}(x)$ $(\alpha \geq -1)$ in a manageable form is established. Meanwhile, the explicit representation of the fundamental polynomials, when they exist, is given. Moreover, it is shown that, if the problem of $(0,1,\cdots,m-2,m)$ interpolation has an infinity of solutions, then the general form of the solutions is $f_0(x)+Cf_1(x)$ with an arbitrary constant C.

1. Introduction

Let us consider a system A of nodes

$$0 \leq x_1 < x_2 < \cdots < x_n, \quad n \geq 2. \tag{1.1}$$

Let $\mathbf{P}_n$ be the set of polynomials of degree at most n and $m \geq 2$ fixed integer. The problem of $(0,1,\cdots,m-2,m)$ interpolation is, given a set of numbers

$$y_{kj}, \quad k \in N := \{1,2,\cdots,n\}, \;\; j \in M := \{0,1,\cdots,m-2,m\}, \tag{1.2}$$

to determine a polynomial $R_{mn-1} \in \mathbf{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k) = y_{kj}, \quad \forall k \in N, \;\; \forall j \in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1} \in \mathbf{P}_{mn-1}$ satisfying (1.3), then we say that the problem of $(0,1,\cdots,m-2,m)$ interpolation on A is regular (otherwise, singular) and $R_{mn-1}(x)$ can be written uniquely as

$$R_{mn-1}(x) = \sum_{\substack{k \in N \\ j \in M}} y_{kj} r_{kj}(x) \tag{1.4}$$

where $r_{kj} \in \mathbf{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k,\nu \in N, \;\; j,\mu \in M \tag{1.5}$$

* Received March 10, 1993.

[1)] The Project Supported by National Natural Science Foundation of China.

and are called the fundamental polynomials. In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \quad k = 1, 2, \cdots, n. \tag{1.6}$$

In [1] and [2] the exact condition of regularity on the parameter $\alpha \geq -1$ of the Laguerre polynomials $L_n^{(\alpha)}(x)$ is found for $(0,2)$ interpolation based on the zeros of these polynomials. The problem of determining the fundamental polynomials is also discussed. But the latter problem is solved for $\alpha = -1$ only. For $\alpha > -1$ the representation of fundamental polynomials is given only in the case when α is an odd integer and only on $(-\infty, 0)$, while a representation on $[0, \infty)$ would be more important. Following the main idea of [1] and [2], in this paper we attempt to give a necessary and sufficient condition of regularity of $(0, 1, \cdots, m-2, m)$ interpolation for the Laguerre abscissas. Meanwhile, we develop a method of finding the explicit representation of the fundamental polynomials when they exist without exception. Thus, our results improve and extend the ones of [1] and [2]. Finally, when the problem of $(0, 1, \cdots, m-2, m)$ interpolation on A is not regular, then for a given set of numbers y_{kj} either there is no polynomial $R_{mn-1}(x)$ satisfying (1.3) or there is an infinity of polynomials with the property (1.3). The possibility of an infinity of solutions raises the question on the dimensionality of their number. We show that in the case of infinitely many solutions the general form of the solutions is

$$R_{mn-1}(x) = f_0(x) + Cf_1(x),$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials and C is an arbitrary number.

2. An Auxiliary Lemma

We first state a lemma given by the author in [3]. To this end we introduce the fundamental polynomials of $(0, 1, \cdots, m-1)$ interpolation. Let $A_{kj},\ B_k \in \mathbf{P}_{mn-1}$ be defined by

$$A_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k, \nu = 1, 2, \cdots, n, \quad j, \mu = 0, 1, \cdots, m-1 \tag{2.1}$$

and

$$B_k(x) := A_{k,m-1}(x) = \frac{1}{m!}(x - x_k)^{m-1} l_k^m(x), \quad k = 1, 2, \cdots, n, \tag{2.2}$$

where

$$l_k(x) := \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \quad \omega_n(x) = c(x - x_1)(x - x_2)\cdots(x - x_n), \quad c \neq 0. \tag{2.3}$$

Then we have

Lemma. *If there is one index i, $1 \leq i \leq n$, such that $\rho_i \in \mathbf{P}_{mn-1}$ with the properties (1.5) exists uniquely, then the problem of $(0, 1, \cdots, m-2, m)$ interpolation is*