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OPTIMUM MODIFIED EXTRAPOLATED JACOBI METHOD FOR CONSISTENTLY ORDERED MATRICES*

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Abstract

This paper is concerned with the investigation of a two-parametric linear stationary iterative method, called Modified Extrapolated Jacobi (MEJ) method, for solving linear systems $Ax = b$, where A is a nonsingular consistently ordered 2-cyclic matrix. We give sufficient and necessary conditions for strong convergence of the MEJ method and we determine the optimum extrapolation parameters and the optimum spectral radius of it, in the case where all the eigenvalues of the block Jacobi iteration matrix associated with A are real. In the last section, we compare the MEJ with other known methods.

1. Introduction and Preliminaries

We consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathrm{R}^{n,n}$, $b \in \mathrm{R}^n$ and $\det(A) \neq 0$. We also assume that A has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.2}$$

where A_{11}, A_{22} are square nonsingular (usually diagonal) matrices. As is know [6], A is a consistently ordered 2-cyclic matrix.

For solving (1.1) we intend to use the following simple iterative method:

$$x^{(m+1)} = L_{\omega_1,\omega_2}x^{(m)} + \Omega D^{-1}b, \quad m = 0, 1, 2, \ldots, \tag{1.3}$$

where

$$\Omega = \begin{bmatrix} \omega_1 I_1 & 0 \\ 0 & \omega_2 I_2 \end{bmatrix}, \tag{1.4}$$

$$D = \operatorname{diag}(A_{11}, A_{22}) \tag{1.5}$$

and

$$L_{\omega_1,\omega_2} = \begin{bmatrix} (1-\omega_1)I_1 & -\omega_1 A_{11}^{-1}A_{12} \\ -\omega_2 A_{22}^{-1}A_{21} & (1-\omega_2)I_2 \end{bmatrix}, \tag{1.6}$$

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In (1.4) and (1.6), ω_1, ω_2 are nonzero parameters (extrapolation parameters) and I_1, I_2 are identity matrices of the same sizes as A_{11} an A_{22} respectively. The construction of method (1.3) is based on the splitting $A = M - N$, where

$$M = D\Omega^{-1} = \Omega^{-1}D \tag{1.7}$$

and therefore, in the sequel, we will call it Modified Extrapolated Jacobi (MEJ) method for the system (1.1). Obviously, for the iteration matrix we have

$$L_{\omega_1,\omega_2} = I - \Omega D^{-1}A = I - \Omega + \Omega T, \tag{1.8}$$

where $T = I - D^{-1}A = L + U$ is the block Jacobi iteration matrix associated with A ($A = D(I - L - U)$, where L and U strictly lower and strictly upper triangular matrices, respectively). It is clear that, if $\omega_1 = \omega_2 = \omega$, then the MEJ method reduces to the known Extrapolated (Block) Jacobi (EJ) or (Block) Jacobi Overrelaxation (JOR) method (see e.g. [5], [8]) for A. It must also be noted that MEJ is a special case of the recently introduced ([2], [3]) Block Modified Accelerated Overrelaxation (MAOR) method, applied to (1.1), which has the form

$$x^{(m+1)} = L_{R,\Omega}x^{(m)} + c, \quad m = 0, 1, 2, \ldots, \tag{1.9}$$

where

$$L_{R,\Omega} = (I - RL)^{-1}[I - \Omega + (\Omega - R)L + \Omega U] = I - (I - RL)^{-1}\Omega D^{-1}A \tag{1.10}$$

and

$$c = (I - RL)^{-1}\Omega D^{-1}b. \tag{1.11}$$

The matrices R and Ω appeared in (1.10)–(1.11) are defined by

$$R = \begin{bmatrix} r_1 I_1 & 0 \\ 0 & r_2 I_2 \end{bmatrix}, \tag{1.12}$$

and (1.4), with r_1, r_2 the acceleration parameters. If $r_1 = r_2 = 0$, then (1.9) reduces to MEJ method.

Our purpose in this paper is to investigate the MEJ method, in order to find sufficient and necessary conditions for strong convergence of it, as well as to determine the optimal values of the extrapolation parameters and the optimal virtual spectral radius (in the sense of [8]) of it, under the further assumption that all the eigenvalues μ of T are real. A basic reason motivating the investigation of MEJ method is the known result that under the above mentioned assumptions the optimum EJ method coincides with the Jacobi method, that is the optimum extrapolation factor is $\omega_{\text{opt}} = 1$. We show that the same is not true for the optimum MEJ method. It must be noted that the obtained results are new, since, as it seems, similar ones are not appeared in the literature, and generalize previous ones related to EJ method. In section 4, we also compare the optimum MEJ method with the following methods: Jacobi, Gauss-Seidel, optimum