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MULTISTEP METHODS FOR A CLASS OF HIGHER ORDER DIFFERENTIAL PROBLEMS: CONVERGENCE AND ERROR BOUNDS*[1)]

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Abstract

In this paper multistep methods for higher order differential systems of the type $Y^{(r)} = f(t, Y)$ are proposed. Such methods permit the numerical solutions of initial value problems for such systems, providing error bounds and avoiding the increase of the computational cost derived from the standard approach based on the consideration of an equivalent extended first order system.

1. Introduction

Higher order differential systems of the form

$$Y^{(r)}(t) = f(t, Y(t)), \quad a \leq t \leq b,$$
$$Y^{(i)}(a) = \Omega_i \in \mathbb{C}^{p\times q}, \quad 0 \leq t \leq r-1, \quad r \geq 2 \tag{1.1}$$

are frequent in a variety of models in physics. These systems arise for example modeling the motion of a system of particles as determined from the laws of classical mechanics such as the interaction of atoms and molecules[2,16,17], the motion of the solar system and space capsules[3,21] and the evolution of star cluster[5]. Other situations where systems of the type (1.1) appear in a natural way may be found in optics[8], quantum theory of scattering[7] or celestial mechanics[3]. Apart from these problems, systems of the type (1.1) arise using the method of lines for solving higher order scalar partial differential systems[20].

(1.1) can be written as an extended first order problem[4]; however, there are advantages in studying methods for problems of the type (1.1) for several reasons:

(a) the transformation of system (1.1) into an extended first order problem increases the computational cost;

(b) the physical meaning of the original magnitudes is lost with the transformation of the system;

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(c) by requiring less generality we may able to produce more efficient algorithms;

(d) useful concepts may be identified, leading to a better understanding of what we require of a numerical method for problems in our chosen class.

Systems of the type (1.1) with $r = 1$ have been treated in [13] for the vector case and in [15] for the matrix case. The special problem

$$Y^{(r)}(t) = f(t), \quad Y^{(i)}(a) = \Omega_i, \quad 0 \le i \le r-1, \quad a \le t \le b$$

has been treated in [14] for the scalar case.

In this paper, we consider problems of the type (1.1) where f is a bounded, continuous function $f : [a,b] \times \mathbb{C}^{p\times q} \to \mathbb{C}^{p\times q}$ satisfying the Lipschitz condition

$$\|f(t,P) - f(t,Q)\| \le L\|P-Q\|, \quad P,Q \in \mathbb{C}^{p\times q}. \tag{1.2}$$

This paper is organized as follows. In section 2 some preliminaries about rational matrix functions are included. In section 3 multistep matrix methods for problems of the type (1.1)–(1.2) are introduced and concepts of consistency, zero-stability and convergence are defined. A family of examples is given. Section 4 deals with the study of the discretization error of multistep methods, in particular it is proved that consistent and zero-stable methods are convergent.

If A is a matrix in $\mathbb{C}^{p\times p}$, we denote by $\|A\|$ its 2-norm, defined in [10]. If B is a matrix in $\mathbb{C}^{p\times q}$, we denote by $\sigma(B)$ the set of all the eigenvalues of B and its spectral radiius $\rho(B)$ is the maximum of the set $\{|z|; z \in \sigma(B)\}$. If $z \in \sigma(B)$, the index of z considered as an eigenvalue of B, denoted by $Ind(z,B)$ is the smallest non-negative integer n such that $Ker(B-zI)^n = Ker(B-zI)^{n+1}$, [6]. The number $Ind(z,B)$ coincides with the dimension of the bigest Jordan block of B in which the eigenvalue z appears in the Jordan canonical form of B. An efficient algorithm for computing $Ind(z,B)$ can be found in [1].

In an analogous way to the definition of matrices of class M, given in [9] , we say that a matrix $B \in \mathbb{C}^{p\times p}$ is of class r if for every eigenvalue $z \in \sigma(B)$ such that $|z| = \rho(B)$, every Jordan block of B associated with z has size $s \times s$ with $s \le r$. Finally, from formulae 0.121 of [11], if q is a positive integer it follows that

$$\sum_{k=1}^{n} k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2}\begin{bmatrix} q \\ 1 \end{bmatrix} B_2 n^{q-1} + \frac{1}{4}\begin{bmatrix} q \\ 3 \end{bmatrix} B_4 n^{q-2} + \frac{1}{6}\begin{bmatrix} q \\ 5 \end{bmatrix} B_6 n^{q-5} + \cdots \tag{1.3}$$

where last term contains either n or n^2 and B_m denotes the m-th Bernouilli number.

2. Preliminaries About Rational Matrix Functions

We begin this section with a result that generalizes lemmas 5.5 and 6.2 of [12].

Theorem 2.1. *Let the polynomial* $p(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_0$ *has only zeros on the unit disk* $|z| \le 1$ *and those with modulus 1 are of multiplicity not exceding*