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CONVERGENCE ACCELERATION OF VECTOR SEQUENCES BY VECTOR PADÉ APPROXIMATION*1)

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Abstract

By making use of vector Padé approximants, a method for accelerating the convergence of vector sequences is derived. The method obtained includes the well known Henrici transformation as a special case. A main character of the method is that it can use partial vector components to accelerate convergence of the whole vector sequence. Results about the efficacy of the method are established. An algorithm and some numerical examples are given.

§1. Introduction

In applied mathematics, we sometimes need to compute vector sequences which, under certain assumptions, converge to the solution of the problem considered. However, it is often the case that the convergence is so slow that the computational approach is of no practical use. Hence, it is natural to employ some convergence acceleration methods. In this paper, we derive a method from the vector Padé approximation introduced in [5]. Differing from the existing acceleration methods (see [3]), our method does not use a fixed number of components of the vectors being accelerated. The number of components used varies with the degrees of Padé approximants.

In order to introduce the vector Padé approximation(VPA), we give firstly the following notations:

$$H_k := \{p(z) : p(z) = \sum_{i=0}^{k} a_i z^i, a_i \in C\},$$

$$E_k := \{e(z) : e(z) = \sum_{i=k+1}^{\infty} a_i z^i, a_i \in C\},$$

$$Z_+^p := \{\vec{n} : \vec{n} = (n_1, \cdots, n_p)^T, n_i \in Z_+, \ i = 1, \cdots, p\},$$

where p is a given positive integer, and Z_+ is the set of all nonnegative integers. $|\vec{n}| = \sum_{i=1}^{p} n_i$, for $\vec{n} \in Z_+^p$.

$$H_{\vec{n}}^p := (H_{n_1}, \cdots, H_{n_p})^T, \quad E_{\vec{w}}^p := (E_{w_1}, \cdots, E_{w_p})^T.$$

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If $g(z) = \sum_{k=0}^{\infty} c_k z^k, c_k \in C$, we denote

$$T^l_{m,n}(g) = \begin{bmatrix} c_l & c_{l-1} & \cdots & c_{l-n+1} \\ c_{l+1} & c_l & \cdots & c_{l-n+2} \\ \vdots & \vdots & & \vdots \\ c_{l+m-1} & c_{l+m-2} & \cdots & c_{l+m-n} \end{bmatrix} \tag{1.1}$$

with the convention $c_k = 0$ if $k < 0$.

Definition. *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k, c_k \in C^p$ be a given power series, $\vec{n} = (n_1, \cdots, n_p)^T \in Z^p_+$ and $\vec{w} = (w_1, \cdots, w_p)^T \in Z^p_+$ be two given integer vectors such that*

$$\vec{e} := \vec{w} - \vec{n} \in Z^p_+, \quad \textit{and} \quad |\vec{w}| = |\vec{n}| + m \, ,$$

where m is a given integer. If we can find a vector polynomial $N(z) \in H^p_{\vec{n}}$ and a scalar polynomial $M(z) \in H_m$ such that

$$f(z)M(z) - N(z) \in E^p_{\vec{w}}, \quad \text{and} \quad M(0) = 1 \, ,$$

then we call $N(z)M(z)^{-1}$ the $[\vec{n}, m, \vec{w}]$ vector Padé approximation of f. We denote it as $[\vec{n}, m, \vec{w}]_f$.

From (4.1) and (4.2) of [5], we have

Theorem 1 (Determinant expression for VPA). *If $H(\vec{n}, m, \vec{w})$ is nonsingular, then*

$$M(z) = \frac{1}{H(\vec{n}, m, \vec{w})} \det \begin{bmatrix} 1 & z & \cdots & z^m \\ B(\vec{n}, m, \vec{w}) & & H(\vec{n}, m, \vec{w}) & \end{bmatrix} , \tag{1.2}$$

$$N(z) = \frac{1}{H(\vec{n}, m, \vec{w})} \operatorname{Det} \begin{bmatrix} f^{(\vec{n})} & z f^{(\vec{n}-1)} & \cdots & z^m f^{(\vec{n}-m)} \\ B(\vec{n}, m, \vec{w}) & & H(\vec{n}, m, \vec{w}) & \end{bmatrix} , \tag{1.3}$$

where

$$H(\vec{n}, m, \vec{w}) = \begin{bmatrix} T^{n_1}_{e_1,m}(f_1) \\ \vdots \\ T^{n_p}_{e_p,m}(f_p) \end{bmatrix}, \quad B(\vec{n}, m, \vec{w}) = \begin{bmatrix} T^{n_1+1}_{e_1,1}(f_1) \\ \vdots \\ T^{n_p+1}_{e_p,1}(f_p) \end{bmatrix} ,$$

$$f^{(\vec{n})}(z) = [f_1^{(n_1)}(z), \cdots, f_p^{(n_p)}(z)]^T, \quad f_i^{(k)}(z) = \begin{cases} \sum_{j=0}^{k} (c_j)_i z^j, & \text{for} \quad k \geq 0 \, , \\ 0, & \text{for} \quad k < 0 \, , \end{cases}$$

for $i = 1, \cdots, p$, and Det $[\cdots]$ *denotes the vector obtained by expanding $[\cdots]$ with respect to its first "row". Here and in the following $(\cdot)_i$ denotes the i-th component of the vector $(\cdot)$.*

In Section 2, we will obtain an acceleration method from VPA and show that the Henrici transformation is a special case of the method. In Section 3 we will use the E-algorithm in [2] to implement it. Some numerical examples are given in Section 4, which show that the VPA method does accelerate the convergence of vector sequences,