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AN INVERSE PROBLEM FOR THE BURGERS EQUATION*

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Abstract

In this paper the Generalized Pulse-Spectrum Technique (GPST) is extended to solve an inverse problem for the Burgers equation. We prove that the GPST is equivalent in some sense to the Newton-Kantorovich iteration method. A feasible numerical implementation is presented in the paper and some examples are excuted. The numerical results show that this procedure works quite well.

§1. Introduction

In this paper, we shall consider an inverse problem of the Burgers equation[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) = f. \tag{1.1}$$

We assume $f \neq 0$ in order to obtain exact solutions which will be used to compare with our numerical results. The coefficient function $\nu = \nu(x)$ has practical senses as conductivity or viscosity, etc[2,3]. Our purpose in the paper is to investigate the identification of the coefficient $\nu(x)$ through the Burgers equation and some initial and boundary conditions. This inverse problem has obviously both theoretical interest and practical importance.

In the next section we shall present an iterative algorithm for identifying numerically the coefficient $\nu(x)$ by using the Generalized Pulse-Spectrum Technique (GPST). The GPST has been applied to many inverse problems and proved to be a versatile and efficient numerical algorithm[4–6]. In Section 3, we consider the same problem as an inverse problem of an abstract operator equation and prove that the algorithm presented in the previous section is equivalent in some sense to the Newton-Kantorovich iterative method. Numerical implementation of the algorithm is discussed in Section 4. Finite difference methods are applied to both the direct and inverse problems, which result in a linear system containing $\Delta\nu$, the improvement of the approximation of ν, as its

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unknown. Some regularization methods are used to treat the ill-posedness of the inverse problem. Several examples are given in Section 5, which show that the numerical algorithm presented in the paper works very well. Finally, a brief discussion of the algorithm and its perfoemance is given.

§2. The Numerical Algorithm

Consider the following initial-boundary value problem of the Burgers equation

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\Big(\nu(x)\frac{\partial u}{\partial x}\Big) = f, \\ u(x,0) = U_0(x), & 0 < x < 1,\ 0 < t \le T. \\ u(0,t) = r_0(t), \quad u(1,t) = r_1(t), \end{cases} \tag{2.1}$$

In order to identify $\nu(x)$ through equation (2.1) we need some auxiliary condition which we assume in the paper as follows

$$Bu(x,t) = \frac{\partial}{\partial x}u(0,t) = r(t). \tag{2.2}$$

The GPST algorithm is used to solve numerically the above inverse problem. First choose a function, say $\nu_0(x)$, as the initial approximation of $\nu(x)$ and then use the procedure described below to obtain the first approximation $\nu_1(x)$.

Suppose that the nth iterative approximation of $\nu(x)$, say $\nu_n(x)$, has been obtained and that $u_n(x,t)$ is the corresponding solution to (2.1) with $\nu(x)$ replaced by $\nu_n(x)$; this means u_n is the solution to the following equation,

$$\begin{cases} \frac{\partial u_n}{\partial t} + u_n\frac{\partial u_n}{\partial x} - \frac{\partial}{\partial x}\Big(\nu_n(x)\frac{\partial u_n}{\partial x}\Big) = f, \\ u_n(x,0) = U_0(x), \\ u_n(0,t) = r_0(t), \quad u_n(1,t) = r_1(t). \end{cases} \tag{2.3}$$

Assume that the $(n+1)$st approximation $\nu_{n+1}(x)$ and the corresponding $u_{n+1}(x,t)$ are as follows

$$\nu_{n+1}(x) = \nu_n(x) + \delta\nu_n(x), \tag{2.4}$$

$$u_{n+1}(x,t) = u_n(x,t) + \delta u_n(x,t). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1) and substracting (2.3) from it, we have

$$\frac{\partial \delta u_n}{\partial t} + u_n\frac{\partial \delta u_n}{\partial x} + \delta u_n\frac{\partial u_n}{\partial x} + \delta u_n\frac{\partial \delta u_n}{\partial x} - \frac{\partial}{\partial x}\Big[\nu_n\frac{\partial \delta u_n}{\partial x} + \delta\nu_n\frac{\partial u_n}{\partial x} + \delta\nu_n\frac{\partial \delta u_n}{\partial x}\Big] = 0.$$

Suppose that the magnitudes of terms $\delta\nu_n, \delta u_n$ and $\partial\delta u_n/\partial x$ are small and their prod-