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SOLVING INVERSE PROBLEMS FOR HYPERBOLIC EQUATIONS VIA THE REGULARIZATION METHOD*

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Abstract

In the paper, we first deduce an optimization problem from an inverse problem for a general operator equation and prove that the optimization problem possesses a unique, stable solution that converges to the solution of the original inverse problem, if it exists, as a regularization factor goes to zero. Secondly, we apply the above results to an inverse problem determining the spatially varying coefficients of a second order hyperbolic equation and obtain a necessary condition, which can be used to get an approximate solution to the inverse problem.

§1. Introduction

Recently, more attention has been paid to various inverse problems for partial differential equations, which arise in a variety of applications such as heat conduction, blood flow in tumors, seismic data inversion, and flow of fluids in porous media. But most inverse problems are ill-posed in the sense of Hadamard. Many of them have no solution, or their solutions, if existing, are not unique. Besides, the solutions of many inverse problems are unstable. Namely, small variations of the data may produce large variations in the solution.

A general inverse problem we consider in the paper is to determine a parameter $q \in Q$, which is a vector-valued function, satisfying the operator equation

$$\Phi(u, q) = f, \tag{1}$$

on the basis of measurement data

$$z = \Lambda u \in \mathcal{K}, \tag{2}$$

where $\Phi \in C^k(Q \times V, F)$, $C^k(X, Y)$ denotes the Banach space of k-times continuously differentiable mappings on X to Y, X and Y are topological spaces, $f \in F$ is given, $u \in V$ is a state of the system (1), and Q, V, F and $\mathcal{K}$ are topological spaces.

The above-mentioned inverse problem usually is ill-posed in the sense of Hadamard.

The regularization method, introduced by Tikhonov [10] for solving Fredholm integral equation, is one of most popular means to solve ill-posed (in a sense of Hadamard)

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problems. Later, Tikhonov applied the method to various ill-posed problems and summarized some of his results in [11]. J. L. Lions applied that method to optimal control problems of distributed parameter systems governed by partial differential equations [9]. Other papers emphasizing the regularization method include [2,3,6,7,9,13].

In §2 we deduce an optimization problem from the inverse problem via a stabilizing functional. We prove that the optimization problem has a unique, stable solution, and that the solution converges to the true solution of the original inverse problem, if it exists, as a regularization factor goes to zero. Therefore, we can take that solution for an approximate solution to the original inverse problem, if it exists, or for a quasisolution to the original inverse problem, if it does not exist owing to some inexactness of the right term f.

In §3 we make an in-depth study of an inverse problem determining the spatially varying coefficients of a second order linear hyperbolic equation to show implementation of the regularization method solving an inverse problem for partial differential equations. First, we prove that the state of the system, which is the solution of the hyperbolic equation, is a smooth function of the parameter q, which is a vector consisting of the coefficients of the hyperbolic equation. Secondly, we make up a smooth functional with a cost functional and a stabilizing functional and then give a necessary condition, which is a variational inequality and can be applied to computation of the approximate solution of the inverse problem.

§2. The General Inverse Problem

We deduce an optimization problem from the above-mentioned general inverse problem. Suppose that $\forall q \in Q_{ad}$, which is a set in a function space to be defined later, we can get a solution to (1), $u = u(q)$, which denotes the dependence of u on q. Consider the cost functional

$$J_{ls}(q) \equiv \|\Lambda u(q) - z\|_{\mathcal{K}}^2, \tag{3}$$

where $\mathcal{K}$ is the observation space. Obviously, $J_{ls}(q) = 0$ if $(u(q), q)$ is the solution to the problem (1)–(2).

Next, we say a nonnegative, continuous functional, $\psi(q)$, is a stabilizing functional, if for any number $r > 0$ the set $\{q \in Q; \psi(q) \le r\}$ is compact.

Now, define the smooth functional

$$J_{sm}(q) \equiv J_{ls}(q) + \beta\psi(q), \quad q \in Q_{ad}, \tag{4}$$

where the regularization factor β is a constant, positive number and $\psi(q)$ is a stabilizing functional. We have

Theorem 1. *Let Q, V, and $\mathcal{K}$ be Banach spaces and suppose that the following assumptions hold:*

H1. *$\forall q \in Q_{ad}$ there is a solution $u = u(q) \in V$ to equation* (1) *and $u \in V$ is continuous in $q \in Q$,*