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L^∞ CONVERGENCE OF CONFORMING FINITE ELEMENTS FOR THE BIHARMONIC EQUATION*[1)]

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Abstract

The paper considers the L^∞ convergence for conforming finite elements, such as Argyris element, Bell element and Bogner-Fox-Schmit element, solving the boundary value problem of the biharmonic equation. The nearly optimal order L^∞ estimates are given.

1. Introduction

The author has considered the L^∞ error estimates of the nonconforming finite elements for the biharmonic equation (see [3]). This paper will discuss the case of conforming finite elements.

Let Ω be a convex polygonal domain. The Dirichlet boundary value problem of the biharmonic equation is the following

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $N = (N_x, N_y)$ is the unit normal of $\partial\Omega$.

For $p \in [1, \infty]$ and $m \geq 0$, let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ be the usual Sobolev spaces, and $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ be the Sobolev norm and semi-norm respectively. When $p = 2$, denote them by $H^m(\Omega)$, $H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

It is known that for $\forall f \in H^{-1}(\Omega)$, problem (1.1) has a unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$, such that

$$\|u\|_{3,\Omega} \leq C\|f\|_{-1,\Omega}, \tag{1.2}$$

with C a positive constant.

Define, for $\forall u, v \in H^2(\Omega)$,

$$a(u,v) = \int\limits_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right) dxdy. \tag{1.3}$$

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Let $f \in L^2(\Omega)$. The variational form of problem (1.1) is to find $u \in H_0^2(\Omega)$, such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $(\cdot,\cdot)$ is the L^2 product.

For $h \in (0, h_0)$ with $h_0 \in (0,1)$, let $\mathcal{T}_h$ be a subdivision of Ω by triangles or rectangle. Let h_T =diam T and ρ_T the largest of the diameters of all circles contained in T. Assume that there exists a positive constant η, independent of h, such that $\eta h < \rho_T < h_T \leq h$ for all $T \in \mathcal{T}_h$. Let $V_h \subset H_0^2(\Omega)$ be a finite element space associated with $\mathcal{T}_h$.

The finite element approximation to problem (1.1) is to find $u_h \in V_h$, such that

$$a(u_h, v) = (f,v), \qquad \forall v \in V_h. \tag{1.5}$$

This paper will show that the estimate of $|u - u_h|_{1,\infty,\Omega}$ is $\mathcal{O}(h^5|\ln h|)$ for Argyris element, $\mathcal{O}(h^4|\ln h|)$ for Bell element and $\mathcal{O}(h^3|\ln h|)$ for Bogner-Fox-Schmit element.

The remaining of the paper is arranged as follows. Section 2 will give the L^∞ estimates for Argyris element and its properties. Section 3 will give the proof of the L^∞ estimate for Argyris element. The last section will consider the case of Bell element and Bogner-Fox-Schmit element.

2. Argyris Element

From now on, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset H_0^2(\Omega)$ be Argyris finite element space associated with $\mathcal{T}_h$. Then $V_h = \{\, v \mid v \in H_0^2(\Omega), v|_T \in P_5(T), \forall T \in \mathcal{T}_h \,\}$. where $P_m(T)$ is the set of all polynomials with degree not greater than m for nonnegative integer m. Denote $Q_m(T)$ as the space consisting of all polynomials with degrees, with respect to x or y, not greater than m.

Let u be a solution of problem (1.1) and u_h that of problem (1.5). If $u \in H_0^2(\Omega) \cap H^6(\Omega)$, the following estimate is true:

$$\|u - u_h\|_{2,\Omega} \leq Ch^4|u|_{6,\Omega}. \tag{2.1}$$

Throughout the paper, C always denotes the positive constant independent of h, with different values in different places. For L^∞ estimates, we have

Theorem 1. *Let V_h be Argyris finite element space, u the solution of problem* (1.1) *and u_h the solution of problem* (1.5). *Then*

$$|u - u_h|_{1,\infty,\Omega} \leq Ch^5|\ln h||u|_{6,\infty,\Omega} \tag{2.2}$$

when $u \in W^{6,\infty}(\Omega)$, and

$$|u - u_h|_{0,\infty,\Omega} \leq Ch^5|\ln h|^{1/2}|u|_{6,\Omega} \tag{2.3}$$

when $u \in H^6(\Omega) \cap H_0^2(\Omega)$.

The proof of Theorem 1 will be given in Section 3. Now we list some properties of Argyris element space.