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THE BIVARIATE SPLINE APPROXIMATE SOLUTION TO THE HYPERBOLIC EQUATIONS WITH VARIABLE COEFFICIENTS*

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§1. Introduction

We are interested in constructing a continuously differentiable surface of approximate solution to the linear hyperbolic equations with variable coefficients. To this end, in Section 2 we develop a finite difference scheme by overlapping three finite difference schemes, which approximate the exact solution and its two partial derivatives. By using this scheme we can conveniently obtain a continuously differentiable surface in the space of bivariate spline functions. In fact, this scheme is determined "uniquely" by the spline space.

In Section 3 we show that the scheme is stable and convergent in L_2-norm. The interesting fact is that, making use of the spline approximation theory, we can estimate the error by using appropriate moduli of smoothness. This leads to the fact that the spline approximate solution and its partial derivatives are convergent to the exact solution and its partial derivatives respectively when the exact solution is in C^2. In fact, in Section 5 we will prove the following result:

Under the hypotheses of Theorem 3.1, if the exact solution v of the hyperbolic equation (2.1) is in $C^2(D)$, then the following estimates hold:

$$\|v-u\|_{D'} \le K\alpha\omega(D^2v,\alpha,D'),$$

$$\left.\begin{array}{l}\|\partial_x(v-u)\|_{D'}\\ \|\partial_t(v-u)\|_{D'}\end{array}\right\} \le K\omega(D^2v,\alpha,D'),$$

where u is a spline approximate solution and $\alpha=\sqrt{h^2+k^2}$, with h and k as steps in the directions x and t respectively.

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The convergence can be shown even if the exact solution is in C^1.

In this paper we handle only the single equation, but it is not difficult to generalize our method to the hyperbolic system. In Sections 4 and 5 we introduce some definitions and results about the space of bivariate spline functions and the bivariate approximation theory. For more details, the readers are referred to [1] and [3]. All constants appearing in this paper are denoted by K although they may be different.

§2. The Finite Difference Scheme

Consider the first order hyperbolic equation of the form

$$\partial_t v = a(x,t)\partial_x v \tag{2.1.1}$$

where $a(x,t) < 0$ for all $(x,t) \in D = [0,+\infty) \times [0,+\infty)$, with the initial-boundary conditions

$$V(x,0) = f(x), \qquad 0 \le x < +\infty; \tag{2.1.2}$$

$$V(0,t) = g(t), \qquad 0 \le t < +\infty. \tag{2.1.3}$$

Let u_i^n, $(\partial_t u)_i^n$ and $(\partial_x u)_i^n$ be three grid functions which approximate the exact solution v, $\partial_t v$ and $\partial_x v$ at the points (ih, nk) respectively, where k is the time step and h is the grid spacing. Those grid functions are determined by the following relations:

$$u_i^n = (1 - a_i^n\lambda)^{-1}[u_i^{n-1} + \frac{k}{2}(\partial_t u_t)_i^{n-1} - \frac{k}{2}a_i^n(\partial_x u)_{i-1}^n - \lambda a_i^n u_{i-1}^n], \tag{2.2.1}$$

$$(\partial_t u)_i^n = -(\partial_t u)_i^{n-1} + 2k^{-1}(u_i^n - u_i^{n-1}), \tag{2.2.2}$$

$$(\partial_x u))_i^n = -(\partial_x u)_{i-1}^n + 2h^{-1}(u_i^n - u_{i-1}^n), \quad i,n = 1,2,3,\cdots, \tag{2.2.3}$$

where $\lambda = k/h$. In order to implement this scheme, let

$$u_0^0 = f(0) = g(0), \tag{2.3.1}$$

$$(\partial_x u)_i^0 = f'(ih), \quad (\partial_t u)_i^0 = a_i^0 f'(ih), \quad i = 0,1,2,\cdots, \tag{2.3.2}$$

$$(\partial_t u)_0^n = g'(nk), \quad (\partial_x u)_0^n = (a_0^n)^{-1}g'(nk), \quad n = 0,1,2,\cdots. \tag{2.3.3}$$

The following lemma shows where the relation (2.2.1) comes from.

Lemma 2.1: *The three relations in (2.2) are equivalent to the relations* (2.2.2), (2.2.3) *and the following relation*

$$(\partial_t u)_i^n = a_i^n(\partial_x u)_i^n, \quad i,n = 0,1,2,3,\cdots. \tag{2.2.1'}$$