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ON THE SENSITIVITY OF SEMISIMPLE MULTIPLE EIGENVALUES*1)

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Abstract

This paper is a supplement to the work in [6] where we investigated the directional derivatives of semisimple multiple eigenvalues of a complex $n \times n$ matrix analytically dependent on several parameters. The result of this paper can be used to define the sensitivity of semisimple multiple eigenvalues in a more resonable way than in [6].

§1. Main Results

This is a supplement to the work in [6]. We shall use the notation described in [5] and [6].

Let $p = (p_1, \cdots, p_N)^T \in \mathbb{C}^N$. Suppose that $A(p) \in \mathbb{C}^{n\times n}$ is an analytic function in a connected open set $\mathcal{B} \in \mathbb{C}^N$. In this paper we consider the eigenproblem

$$A(p)x(p) = \lambda(p)x(p), \quad \lambda(p) \in \mathbb{C}, \quad x(p) \in \mathbb{C}^n, \quad p \in \mathcal{B}. \tag{1.1}$$

Let $\mu(p)$ be a function defined in $\mathcal{B}$. The directional derivative of $\mu(p)$ at $p^* \in \mathcal{B}$ in the direction ν, denoted by $D_\nu\mu(p^*)$, is defined as follows:

$$D_\nu\mu(p^*) \equiv \lim_{\tau\to+0} \frac{\mu(p^*+\tau\nu)-\mu(p^*)}{\tau}, \tag{1.2}$$

where $\nu \in \mathbb{C}^N$ with $\|\nu\|_2 = 1$ and τ is a positive parameter.

Without loss of generality, we may investigate the directional derivatives of the eigenvalues of $A(p)$ at the origin of $\mathbb{C}^N$. The following two theorems are the main results of this paper.

Theorem 1.1. *Let $A(p) \in \mathbb{C}^{n\times n}$ be an analytic function of $p = (p_1, p_2, \cdots, p_N)^T$ in some neighbourhood $\mathcal{B}(0)$ of the origin of $\mathbb{C}^N$, and let $\lambda(A(p)) = \{\lambda_s(p)\}_{s=1}^n$ for $p \in \mathcal{B}(0)$, in which $\lambda_1(0) = \cdots = \lambda_r(0) = \lambda_1$. Suppose that there are matrices $X, Y \in \mathbb{C}^{n\times n}$ satisfying*

$$X = (X_1, X_2), \quad Y = (Y_1, Y_2), \quad X_1, Y_1 \in \mathbb{C}^{n\times r}, \quad X^HY = I \tag{1.3}$$

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and

$$Y^H A(0) X = \begin{pmatrix} \lambda_1 I^{(r)} & 0 \\ 0 & A_2 \end{pmatrix}, \quad \lambda_1 \bar{\in} \lambda(A_2). \tag{1.4}$$

Then, for any fixed direction $\nu \in \mathbb{C}^N$ *with* $\|\nu\|_2 = 1$, *there are a positive scalar* β *and* r *single-valued continuous functions* $\mu_1(\tau\nu), \cdots, \mu_r(\tau\nu)$ *for* $\tau \in [-\beta, \beta]$ *such that* $\{\mu_s(\tau\nu)\}_{s=1}^r$ *are* r *of the eigenvalues of* $A(\tau\nu)$, *the set* $\{\mu_s(\tau\nu)\}_{s=1}^r$ *and the set* $\{\lambda_s(\tau\nu)\}_{s=1}^r$ *are identical, and there is a one-to-one correspondence between the elements of the two sets. Moreover, we have*

$$\{D_\nu \mu_s(0)\}_{s=1}^r = \lambda\Big(\sum_{j=1}^N \nu_j Y_1^H \Big(\frac{\partial A(p)}{\partial p_j}\Big)_{p=0} X_1\Big). \tag{1.5}$$

Theorem 1.2. *Let* $A(p), B(0), \lambda_1(p), \cdots, \lambda_r(p), \lambda_1, X, Y$ *and* ν *be described as in Theorem* 1.1. *Define*

$$s_p^{(\nu)}(\lambda_1) \equiv \lim_{\tau \to 0} \max_{\substack{z \in \mathbb{C} \\ |z| = \tau > 0}} \frac{\max_{1 \le j \le r} |\lambda_j(z\nu) - \lambda_1|}{|z|}. \tag{1.6}$$

Then

$$s_p^{(\nu)}(\lambda_1) = \rho\Big(\sum_{j=1}^N \nu_j Y_1^H \Big(\frac{\partial A(p)}{\partial p_j}\Big)_{p=0} X_1\Big), \tag{1.7}$$

where $\rho(\cdot)$ *denotes the spectral radius of a matrix.*

Remark 1.3. The difficulty in investigating the local behaviours of a semisimple multiple eigenvalue of multiplicity $r > 1$ lies in that the r eigenvalues, as functions of some parameters, may have singularity at the intersection point (ref [2, p.74–76]). Even if in the case of one complex parameter z, the r eigenvalues are in general not continuous in any neighbourhood of the singular point (ref. [2, p.125]). Fortunately, Kato [2, p.125–127] proved that, if the simple parameter z changes over an interval $[\alpha, \beta]$ of the real line, then there exist r single-valued continuous functions, the values of which constitute the set of r eigenvalues for each $z \in [\alpha, \beta]$. Therefore we may take the r single-valued continuous functions as the r eigenvalues for $z \in [\alpha, \beta]$. This fact is just one of the keys to investigating the directional derivatives of semisimple multiple eigenvalues in this paper.

Remark 1.4. M. Overton and R. Womersley [4] discussed directional derivatives of semisimple multiple eigenvalues of the matrix

$$A_0 + \sum_{k=1}^m \xi_k A_k,$$

where $\{A_k\}$ are given real $n \times n$ matrices, and $\{\xi_k\}$ are real parameters.

We shall give the proofs of Theorem 1.1 and Theorem 1.2 in §2 and give some applications in §3.