Journal of Computational Mathematics, Vol.9, No.1, 1991, 9–16.

A SEQUENTIAL ALGORITHM FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS*1)

Fang Kai-tai
(*Institute of Applied Mathematics, Academia Sinica, Beijing, China*)

Wang Yuan
(*Institute of Mathematics, Academia Sinica, Beijing, China*)

Abstract

A sequential algorithm for solving a system of nonlinear equations based on the number-theoretic method is proposed. In order to illustrate the effectiveness of the method, the following two problems are discussed in detail: the problems for finding out a representative point of a continuous univariate distribution, and a fixed point of a continuous mapping of a closed bounded domain into itself.

§1. Introduction

Suppose D is a domain of $\boldsymbol{R}^s$. We want to solve the system of equations

$$\begin{cases} f_1(\boldsymbol{x}) = f_1(x_1, \cdots, x_s) = 0, \\ \quad \cdots\cdots \\ f_t(\boldsymbol{x}) = f_t(x_1, \cdots, x_s) = 0. \end{cases} \tag{1.1}$$

There are many well-known methods for solving (1.1) if f_i's are all linear, but it is difficult to find out an analytic expression for the solutions of (1.1) in usual if f_i's are not all linear functions, so that (1.1) can be solved only by numerical methods, for example (see [5]), the iteration method (see [1]), Newton's method (see [6]), Brown's method[3], Brent's method[2], quasi Newton's method (see [5]), etc. However, the above methods are contained in detail in a book of Feng [6]. These methods require that f_i's have continuous derivatives of first order or even higher orders, or satisfy certain properties of convexity in order that the convergences of these methods are ensured. It is difficult to obtain the explicit formulas of derivatives of the functions f_i's, and sometimes f_i's even do not satisfy the required conditions, for instance, max, min and $|x|$ appear in the expressions of f_i's.

In fact the problem for solving the system of equations (1.1) can be reduced to a problem of optimization. Let

$$L(\boldsymbol{x}) = \sum_{i=1}^{t} |f_i(\boldsymbol{x})|, \quad \boldsymbol{x} \in D \tag{1.2}$$

* Received February 5, 1990.

1) The Project Supported by National Natural Science Foundation of China.

or

$$\tilde{L}(x) = \sum_{i=1}^{t} f_i^2(x), \quad x \in D. \tag{1.3}$$

Then the problem of finding out a solution $x_0 = (x_{01}, \cdots, x_{0s})$ of (1.1) is equivalent to the problem of finding out a point such that $L(x)$ (or $\tilde{L}(x)$) attains its minimum. Notice that such points that $L(x)$ (or $\tilde{L}(x)$) attains its minimum $M = 0$ are not unique in general, and our aim is to find out at least one among them.

We have proposed a sequential algorithm for optimization based on the number-theoretic method, and it is denoted by SNTO [9]. The continuities are required only for the functions f_i's in SNTO such that the convergences of the approximate minimum M^* and the maximum point x^* to the respective M and x_0 are ensured. Besides, it is easy to work out a program in SNTO, and more precisely the programs are almost the same for distinct sets of f_i's. It is the aim of our paper to recommend SNTO for finding an approximate minimum point of L (or $\tilde{L}$), that is, an approximate solution of (1.1). In Section 2, we give SNTO in detail. In order to illustrate the effectiveness and also universality of SNTO, we apply SNTO to treat two problems. The first is the so-called quantization problem. Let X be a random variable with a continuous cumulative distribution function $F(x)$ with a standard deviation 1 and n be a given positive integer. For any given numbers $-\infty < x_1 < x_2 < \cdots < x_n < \infty$, an n-level quantizer Q_n is defined by

$$Q_n(x) = x_k, \quad \text{if } a_k < x \le a_{k+1}, \quad k = 1, \cdots, n,$$

where

$$a_1 = -\infty, \quad a_{n+1} = \infty, \quad a_k = (x_k + x_{k-1})/2, \quad k = 2, \cdots, n.$$

We use the mean square error (MSE)

$$\text{MSE}(x) = E(X - Q_n(X))^2 = \int_{-\infty}^{\infty} \min(x - x_i)^2 p(x) dx \tag{1.4}$$

to measure the distortion between X and $Q_n(X)$, where $x = (x_1, \cdots, x_n)$ and $p(x)$ denotes the probability density function (pdf) of $F(x)$. We shall call x^* a representative point of $F(x)$ if it has the least MSE, i.e. $\text{MSE}(x^*) = \min_x \text{MSE}(x)$. The problem of finding out a representative point appears in many fields, such as information theory, clustering analysis, theory of quantization and theory of stochastic simulation. Max[12], Lloyd[11], and Fang and He[7] proposed independently numerical methods for finding out the representative points. Their methods are the same in essence, where as Fang and He's method is to reduce the above problem to a problem of solving a system of nonlinear equations (cf. (2.2)). In this paper, we shall give a numerical method for finding out a representative point based on the number-theoretic method.

Let f be a continuous mapping which maps a closed bounded domain D into itself. We shall call x_0 a fixed point of f if $f(x_0) = x_0$. Our second problem is to find out a fixed point of f. This problem is close to the problem of solving a system of nonlinear equations, and there appeared several related monographs in recent years, for instances, [13], [14] and [15].

Let

$$L(x) = ||x - f(x)||, \quad x \in D, \tag{1.5}$$

where $||\cdot||$ denotes l_1 or l_2 modulus. Then the problem of finding out a fixed point of $f(x)$ is reduced to the problem of finding out a point $x = x_0$ of D such that $L(x)$ attains its minimum at x_0. Hence we may use SNTO also to find out an approximate fixed point.