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A SPECTRAL-DIFFERENCE SCHEME FOR THREE-DIMENSIONAL VORTICITY EQUATIONS WITH SINGLE PERIODICAL BOUNDARY CONDITION*1)

Guo Ben-yu
(*Shanghai University of Science and Technology, Shanghai, China*)

Abstract

We develop a spectral-difference scheme to solve three-dimensional vorticity equation with single periodical boundary condition. We prove the conservation, generalized stability and convergence. The numerical experiments show that this scheme gives much better results than usual difference schemes.

§1. Introduction

Let $x = (x_1, x_2, x_3)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $\Omega = Q \times I$, where $Q = \{(x_1,x_2), 0 < x_1, x_2 < 1\}$ and $I = \{x_3, 0 < x_3 < 2\pi\}$. Let $\Gamma_p^- = \{x \in \bar{\Omega}, x_p = 0\}$, $\Gamma_p^+ = \{x \in \bar{\Omega},\ x_p = 1\}$ $(p = 1,2)$, $\Gamma_p = \Gamma_p^- \cup P_p^+$ and $\Gamma = \Gamma_1 \cup \Gamma_2$.

Let $\xi(x,t)$ and $\psi(x,t)$ denote the vorticity vector and the stream vector respectively. $\xi_0(x)$ and $f_l(x,t)(l = 1,2)$ are given functions. Their components are denoted by $\xi^{(p)}(x,t)$, $\psi^{(p)}(x,t)$, $\xi_0^{(p)}(x,t)$ and $f_l^{(p)}(x,t)$, $p = 1,2,3$. We consider the three-dimensional vorticity equation as follows:

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + [(\nabla \times \psi)\cdot]\xi - (\xi \cdot \nabla)(\nabla \times \psi) - \nu \nabla^2 \xi = f_1, & (x,t) \in \Omega \times (0,T], \\ -\nabla^2 \psi = \xi + f_2, & (x,t) \in \Omega \times [0,T], \\ \xi(x,0) = \xi_0(x), & x \in \bar{\Omega} \end{cases} \tag{1.1}$$

where ν is a positive constant and $f_l(x,t)$ and $\xi_0(x)$ have the period 2π for the variable x_3.

There are many papers concerning the finite difference methods for solving (1.1) [1,2]. The author and others[3,4] proposed spectral and pseudospectral methods to solve the periodical problem. Because problem (1.1) is only periodical for the variable x_3, we cannot use the method in [3, 4]. Recently, the author[5,6] developed a spectral-difference method to solve such partially periodical problems. In this paper, we propose a spectral-difference method for (1.1).

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1) The computation was done by Mr. Xiong Yue-shan.

§2. Notations and the Scheme

Let h and τ be the mesh sizes of the variables $x_p (p = 1, 2)$ and $t, Mh = 1$. Define

$$
\begin{aligned}
&Q_h = \{(x_1, x_2) = (j_1h, j_2h)/1 \le j_1, j_2 \le M-1\}, && \bar{Q}_h = \{(j_1h, j_2h)/0 \le j_1, j_2 \le M\},\\
&\Omega_h = Q_h \times I, && \bar{\Omega}_h = \bar{Q}_h \times I,\\
&\Gamma_{h,p}^- = \{x \in \bar{\Omega}_h/x_p = 0\}, && \Gamma_{h,p}^+ = \{x \in \bar{\Omega}_h/x_p = 1\}, \quad p = 1, 2,\\
&\Gamma_{h,p} = \Gamma_{h,p}^- \cup \Gamma_{h,p}^+, && \Gamma_h = \Gamma_{h,1} \cup \Gamma_{h,2},\\
&\Omega_{h,p}^{*-} = \{x \in \Omega_h/x_p = h\}, && \Omega_{h,p}^{*+} = \{x \in \Omega_h/x_p = 1-h\}, \quad p = 1, 2,\\
&\Omega_{h,p}^* = \Omega_{h,p}^{*-} \cup \Omega_{h,P}^{*,+}, && \Omega_h^* = \Omega_{h,1}^* \cup \Omega_{h,2}^*,\\
&S_\tau = \{t = k\tau/k = 0, 1, 2, \cdots, \}.
\end{aligned}
$$

Let $u(x, t)$ and $v(x, t)$ be the three-dimensional vectors. We define

$$u_{x_p}(x,t) = \frac{1}{h}(u(x+he_p, t) - u(x,t)), \quad u_{\bar{x}_p}(x,t) = u_{x_p}(x - he_p, t), \quad p = 1, 2,$$

$$u_{\hat{x}_p}(x,t) = \frac{1}{2}(u_{x_p}(x,t) + u_{\bar{x}_p}(x,t)), \quad u_n(x,t) = \begin{cases} -u_{x_p}(x,t), & \text{for } x \in \Gamma_{h,p}^-,\\ u_{\bar{x}_p}(x,t), & \text{for } x \in \Gamma_{h,p}^+,\end{cases}$$

$$\Delta u(x,t) = u_{x_1\bar{x}_1}(x,t) + u_{x_2\bar{x}_2}(x,t) + \frac{\partial^2 u}{\partial x_3^2}(x,t), \quad u_t(x,t) = \frac{1}{\tau}(u(x, t+\tau) - u(x,t)).$$

We define the following scalar products and norms:

$$
\begin{aligned}
&(u(x_1,x_2,t), v(x_1,x_2,t))_I = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\bar{v}(x,t)dx_3,\\
&\|u(x_1,x_2,t)\|_I^2 = (u(x_1,x_2,t), u(x_1,x_2,t))_I,\\
&(u(x_3,t), v(x_3,t))_{Q_h} = h^2 \sum_{(x_1,x_2)\in Q_h} u(x,t)\bar{v}(x,t),\\
&\|u(x_3,t)\|_{Q_h}^2 = (u(x_3,t), u(x_3,t))_{Q_h},\\
&(u(t), v(t)) = h^2 \sum_{(x_1,x_2)\in Q_h} (u(x_1,x_2,t), v(x_1,x_2,t))_I,\\
&\|u(t)\|^2 = (u(t), u(t)), \quad \|u(t)\|_{l^4}^4 = \||u(t)|^2\|^2,\\
&|u(t)|_1^2 = \frac{1}{2}\sum_{p=1,2}(\|u_{x_p}(t)\|^2 + \|u_{\bar{x}_p}(t)\|^2) + \left\|\frac{\partial u}{\partial x_3}(t)\right\|^2,\\
&\|u(t)\|_1^2 = \|u(t)\|^2 + |u(t)|_1^2,\\
&|u(t)|_2^2 = \frac{1}{2}\sum_{p=1,2}(|u_{x_p}(t)|_{1,\Omega_h/\Omega_h^*} + |u_{\bar{x}_p}(t)|_{1,\Omega_h/\Omega_h^*}) + \left|\frac{\partial u}{\partial x_3}(t)\right|_1^2,\\
&\|u(t)\|_2^2 = \|u(t)\|_1^2 + |u(t)|_2^2,
\end{aligned}
$$