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RAYLEIGH QUOTIENT AND RESIDUAL OF A DEFINITE PAIR*1)

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Abstract

Let $\{A, B\}$ be a definite matrix pair of order n, and let Z be an l-dimensional subspace of $\mathbb{C}^n$. In this paper we introduce the Rayleigh quotient matrix pair $\{H_1, K_1\}$ and residual matrix pair $\{R_A, R_B\}$ of $\{A, B\}$ with respect to Z, and use the norm of $\{R_A, R_B\}$ to bound the difference between the eigenvalues of $\{H_1, K_1\}$ and that of $\{A, B\}$, and to bound the difference between Z and an l-dimensional eigenspace of $\{A, B\}$. The corresponding classical theorems on the Hermitian matrices can be derived from the results of this paper.

§1. Preliminaries

Notation. $\mathbb{C}^{m\times n}$: the set complex $m \times n$ matrices, and $\mathbb{C}^n = \mathbb{C}^{n\times 1}$; $\mathbb{C}_r^{m\times n}$: the set of matrices with rank r in $\mathbb{C}^{m\times n}$. $\mathbb{R}$: the set of real numbers. $\bar{A}$: the conjugate of A. A^T: the transpose of A. $A^H = \bar{A}^T$. $A^\dagger$: the Moore-Penrose inverse of a matrix A. $\lambda(A)$: the set of the eigenvalues of A. $\mathcal{R}(X_1)$: the column space of a matrix X_1. $\|\cdot\|_2$: the Euclidean norm for vectors and the spectral norm for matrices. $\|\cdot\|_F$: the Frobenius matrix norm.

In this section we give some definitions and basic results on definite pairs.

Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian. The matrix pair $\{A, B\}$ is a definite pair if [3]–[6]

$$c(A, B) \equiv \min_{\|x\|_2=1} \left|x^H(A + iB)x\right| > 0, \tag{1.1}$$

where $i = \sqrt{-1}$. The set of all definite pairs of order n will be denoted by $\mathbb{D}(n)$.

Let $\{A, B\} \in \mathbb{D}(n)$. A non-zero vector $x \in \mathbb{C}^n$ is an eigenvector of $\{A, B\}$ belonging to the eigenvalue (α, β), if

$$(\alpha, \beta) \neq (0, 0), \quad \beta Ax = \alpha Bx.$$

The set of the eigenvalues of $\{A, B\}$ will be denoted by $\lambda(A, B)$.

Let $\{A, B\} \in \mathbb{D}(n)$, and let $\mathcal{X} = \mathcal{R}(X_1)$ be an l-dimensional subspace of $\mathbb{C}^n$, in which $X_1 \in \mathbb{C}_l^{n\times l}$. $\mathcal{X}$ is called an eigenspace of $\{A, B\}$ if[3]

$$\dim\,(A\mathcal{X} + B\mathcal{X}) \leq \dim\,(\mathcal{X}).$$

Let $\mathcal{X} = \mathcal{R}(X_1)$, $X_1 \in \mathbb{C}_l^{n\times l}$. It is easy to prove that the following statements are equivalent (ref. [3]–[5]):

1) $\mathcal{X}$ is an l-dimensional eigenspace of $\{A, B\}$;

2) $\mathcal{X}$ is spanned by l linearly independent eigenvectors of $\{A, B\}$;

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3) there is an l-dimensional subspace $\mathcal{Y} \subseteq \mathbb{C}^n$ such that

$$A\mathcal{X}, \quad B\mathcal{X} \subseteq \mathcal{Y};$$

4) there is $\{A', B'\} \in \mathbb{D}(l)$ such that

$$AX_1B' = BX_1A';$$

5) there are $Y_1 \in \mathbb{C}_l^{n\times l}$ and $\{A_1, B_1\} \in \mathbb{D}(l)$ such that $Y_1^HX_1 = I$, and

$$AX_1 = Y_1A_1, \quad BX_1 = Y_1B_1. \tag{1.2}$$

Now we introduce the Rayleigh quotient and the residual of a definite pair.

Definition 1.1. *Let $\{A, B\} \in \mathbb{D}(n), Z_1 \in \mathbb{C}^{n\times l}$, and $Z_1^HZ_1 = I$. Let*

$$H_1 = Z_1^HAZ_1, \quad K_1 = Z_1^HBZ_1. \tag{1.3}$$

Then $\{H_1, K_1\}$ is called the Rayleigh quotient matrix pair (or simply, the Rayleigh quotient) of $\{A, B\}$ with respect to Z_1.

Suppose that $\mathcal{R}(X_1)$ is an l-dimensional eigenspace of $\{A, B\} \in \mathbb{D}(n)$. From 5), there are $Y_1 \in \mathbb{C}_l^{n\times l}$ and $\{A, B\} \in \mathbb{D}(l)$ such that $Y_1^HX_1 = I$, and the relations (1.2) hold. From (1.2) we get

$$A_1 = X_1^HAX_1, \quad B_1 = X_1^HBX_1$$

and

$$Y_1 = (AX_1A_1 + BX_1B_1)(A_1^2 + B_1^2)^{-1}.$$

This suggests the following definition.

Definition 1.2. *Let $\{A, B\} \in \mathbb{D}(n)$. For any fixed $Z_1 \in \mathbb{C}^{n\times l}$ satisfying $Z_1^HZ_1 = I$, construct H_1 and K_1 by (1.3), and let*

$$W_1 = (AZ_1H_1 + BZ_1K_1)(H_1^2 + K_1^2)^{-1} \tag{1.4}$$

and

$$R_A(Z_1) = AZ_1 - W_1H_1, \quad R_B(Z_1) = BZ_1 - W_1K_1. \tag{1.5}$$

Then $\{R_A(Z_1), R_B(Z_1)\}$ is called the residual matrix pair (or simply, the residual) of $\{A, B\}$ with respect to Z_1.

It is easy to see that if $Z_1 \in \mathbb{C}^{n\times l}$ and $Z_1^HZ_1 = I$, then $\mathcal{R}(Z_1)$ is an eigenspace of $\{A, B\} \in \mathbb{D}(n)$ if and only if

$$R_A(Z_1) = 0, \quad R_B(Z_1) = 0.$$

We have proved in [6] that the precision of the eigenvalues of $\{H_1, K_1\}$ as l approximate eigenvalues of $\{A, B\}$ is higher than that of $\mathcal{R}(Z_1)$ as its approximate eigenspace. In this paper we shall use the norm of $\{R_A(Z_1), R_B(Z_1)\}$ to bound the difference between the eigenvalues of $\{H_1, K_1\}$ and that of $\{A, B\}$ to bound the difference between $\mathcal{R}(Z_1)$ and an l-dimensional eigenspace of $\{A, B\}$. From the results of this paper one can derive the corresponding classical theorems on Hermitian matrices.

At the end of this section we cite two perturbation theorems which will be used in the following.

Theorem 1.3[3]. *Let $\{A, B\}, \{\tilde{A}, \tilde{B}\} \in \mathbb{D}(n)$, and let $\lambda(A, B) = \{(\alpha_i, \beta_i)\}$, $\lambda(\tilde{A}, \tilde{B}) = \{(\tilde{\alpha}_i, \tilde{\beta}_i)\}$. Then there is a permutation π of $\{1, \cdots, n\}$ such that*

$$\rho\big((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})\big) \le \frac{\sqrt{\|\tilde{A} - A\|_2^2 + \|\tilde{B} - B\|_2^2}}{c(A, B)}, \quad i = 1, \cdots, n.$$