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THE CONVEXITY OF FAMILIES OF ADJOINT PATCHES FOR A BEZIER TRIANGULAR SURFACE

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Abstract

A necessary and sufficient condition for the convexity of adjoint patches for a Bezier triangular surface is presented. Furthermore, it is proved that this condition is equivalent to the fact that the adjoint patches form a decreasing sequence as the corresponding degree decreases. The condition can be easily computationally verified.

§1. Introduction

Consider a given triangle $T \subset R^2$. A Bernstein-Bezier surface over T is usually expressed as

$$B^n := B^n(p) := \sum_{i+j+k=n} f_{i,j,k} J^n_{i,j,k}(p) \tag{1}$$

where

$$J^n_{i,j,k}(p) := \frac{n!}{i!\, j!\, k!} u^i v^j w^k,$$

$p := (u, v, w) \in T$ is a point given by its barycentric coordinates, and $F := \{f_{i,j,k} \in R \mid i+j+k = n,\ i,j,k \geq 0\}$ is a set of prescribed real numbers. The de Casteljau algorithm[1] provides a stable and efficient tool for the evaluation of $B^n(p)$. It is well known that it has also a simple geometric interpretation, i.e. it can be viewed as a sequence of plain interpolations. To be precise, let us follow [2] and define partial shift operators:

$$E_1 g_{i,j,k} := g_{i+1,j,k}, \quad E_2 g_{i,j,k} := g_{i,j+1,k}, \quad E_3 g_{i,j,k} := g_{i,j,k+1}. \tag{2}$$

Let the nodes $P_{i,j,k}$ that correspond to $f_{i,j,k}$ be given by

$$P_{i,j,k} := (i/n,\ j/n,\ k/n), \quad i+j+k = n.$$

For a given $P \in T$, the de Casteljau algorithm computes the values

$$\begin{aligned} f^m_{i,j,k} := f^m_{i,j,k}(p) &:= (uE_1 + vE_2 + wE_3)^m f_{i,j,k} \\ &= \sum_{\substack{\alpha+\beta+\gamma=m \\ i+j+k=n-m}} f_{i+\alpha,j+\beta,k+\gamma} J^m_{\alpha,\beta,\gamma}(p), \end{aligned} \tag{3}$$

that correspond to the nodes

$$P^m_{i,j,k} := P^m_{i,j,k}(p) := (uE_1 + vE_2 + wE_3)^m P_{i,j,k} = \left(\frac{i+mu}{n}, \frac{j+mv}{n}, \frac{k+mw}{n}\right). \tag{4}$$

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In particular, $f^n_{0,0,0}$ is the value of B^n at the point $P^n_{0,0,0} = P$. Let $F^m := \{f^m_{i,j,k}\}$ and consider $P^m_{i,j,k}$. These nodes belong to a smaller triangle with vertices

$$P^m_{n-m,0,0}, \quad P^m_{0,n-m,0}, \quad P^m_{0,0,n-m}.$$

We denote it by T^m. Quite obviously, F^m and $\{P^m_{i,j,k}\}$ depend on the point $P \in T$ at which we are evaluating B^n. Nevertheless, we can adjoin to F^m an $(n-m)$th degree Bezier surface over T_m. In barycentric coordinates with respect to T^m it reads

$$B_p^{n-m} := \sum_{i,j+k=n-m} f^m_{i,j,k} J^{n-m}_{i,j,k}. \tag{5}$$

It is called $(n-m)$th adjoint patch of B^n (for the given point p). In [4] it is shown that the original surface B^n is an envelope of the family $\{B_p^m\}$. This explains why the study of adjoint patches could be useful. In the next section we shall discuss the convexity of families of adjoint patches and provide a simple necessary and sufficient condition.

§2. Convexity of Adjoint Patches

In [4] the following conclusion was proved: If the inequalities

$$\begin{aligned} D_1 f_{i,j,k} &:= (E_1 - E_2)(E_1 - E_3) f_{i,j,k} \geq 0, \\ D_2 f_{i,j,k} &:= (E_2 - E_1)(E_2 - E_3) f_{i,j,k} \geq 0, \\ D_3 f_{i,j,k} &:= (E_3 - E_1)(E_3 - E_2) f_{i,j,k} \geq 0 \end{aligned} \tag{6}$$

hold for $i+j+k = n-2$, then the adjoint patch B_p^{n-m} is convex over T^m, for all $m = 1, 2, \cdots, n$. The condition (6) is only sufficient, not necessary. We proceed with a necessary and sufficient condition that can be easily verified.

Theorem 1. *The adjoint patch B_p^{n-m} is convex over $T^m, m = 1, 2, \cdots, n$, for any $P \in T$ if and only if the data F satisfy*

$$\begin{aligned} &(D_1 + D_2) f_{i,j,k} \geq 0, \quad (D_2 + D_3) f_{i,j,k} \geq 0, \quad (D_1 + D_3) f_{i,j,k} \geq 0, \\ &D_1 f_{i,j,k} D_2 f_{i,j,k} + D_1 f_{i,j,k} D_3 f_{i,j,k} + D_2 f_{i,j,k} D_3 f_{i,j,k} \geq 0 \end{aligned} \tag{7}$$

for all $i+j+k = n-2$.

Proof. The conditions (7) imply the convexity of B^n over $T^{[3]}$. But any B_p^{n-m} is also a Bezier surface corresponding to F^m. Therefore, it is sufficient to prove that (7) hold for any F^m. Assume that F^m satisfies (7) for some fixed $m \geq 0$. We obtain by (3) and (6)

$$D_1 f^{m+1}_{i,j,k} = D_1(uE_1 + vE_2 + wE_3) f^m_{i,j,k} = uD_1 f^m_{i+1,j,k} + vD_1 f^m_{i,j+1,k} + wD_1 f^m_{i,j,k+1} \tag{8}$$

and similar equalities for $D_2 f^{m+1}_{i,j,k}, D_3 f^{m+1}_{i,j,k}$. Thus by assumption on F^m,

$$(D_1 + D_2) f^{m+1}_{i,j,k} = [u(D_1 + D_2) f^m_{i+1,j,k} + v(D_1 + D_2) f^m_{i,j+1,k} + w(D_1 + D_2) f^m_{i,j,k+1}] \geq 0, \tag{9}$$

and

$$(D_1 + D_3) f^{m+1}_{i,j,k} \geq 0, \tag{10}$$

$$(D_2 + D_3) f^{m+1}_{i,j,k} \geq 0 \tag{11}$$

for all $i+j+k = n-(m+1)-2$.