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ON NUMERICAL SOLUTION OF QUASILINEAR BOUNDARY VALUE PROBLEMS WITH TWO SMALL PARAMETERS*

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Abstract

We consider the singular perturbation problem

$$-\varepsilon^2 u'' + \mu b(x,u)u' + c(x,u) = 0, \quad u(0), u(1) \text{ given},$$

with two small parameters ε and $\mu, \mu = \varepsilon^{1+p}, p > 0$. The problem is solved numerically by using finite difference schemes on the mesh which is dense in the boundary layers. The convergence uniform in ε is proved in the discrete L^1 norm. Some convergence results are given in the maximum norm as well.

§1. Introduction

Consider the following singularly perturbed boundary value problem:

$$Tu := -\varepsilon^2 u'' + \mu b(x,u)u' + c(x,u) = 0, \quad x \in I := [0,1], \tag{1.1a}$$

$$Bu := (u(0), u(1)) = (U_0, U_1), \tag{1.1b}$$

where ε is a small parameter:

$$0 < \varepsilon \le \varepsilon^* \ll 1,$$

and

$$\mu = \varepsilon^{1+p}, \quad p > 0.$$

U_0, U_1 are given numbers. We suppose that functions b and c are sufficiently smooth and

$$c_u(x,u) > c_* > 0, \quad x \in I, \quad u \in R. \tag{1.2}$$

This implies that there exist numbers u^* and u_* such that

$$c(x,u_*) < 0 < c(x,u^*), \quad x \in I, \quad u \in R, \tag{1.3a}$$

$$u_* \le U_j \le u^*, \quad j = 0,1. \tag{1.3b}$$

This means that u^* and u_* are upper and lower solutions, respectively, to problem (1.1). Hence (1.1) has a solution, which will be denoted by u_ε. Moreover,

$$u_\varepsilon(x) \in W := [u_*, u^*], \quad x \in I. \tag{1.4}$$

We shall use the conservation form of equation (1.1a):

$$Tu = -\varepsilon^2 u'' + \mu f(x,u)' + g(x,u) = 0, \quad x \in I, \tag{1.5}$$

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where

$$f(x,u) = \int_{u_*}^{u} b(x,s)ds, \quad g(x,u) = c(x,u) - \mu f_x(x,u).$$

Throughout the paper we shall assume that ε^* is sufficiently small. Then,

$$g_u(x,u) \geq g_* > 0, \quad x \in I, \quad u \in W.$$

Hence, because of the inverse monotonicity of the operator (T, B), u_ε is the unique solution satisfying (1.4); see [4].

Problems of type (1.1) belong to the class of two-parameter problems. The asymptotic behaviour of linear two-parameter problems was investigated in [7], and semilinear (i.e. $b = b(x)$) problems were treated numerically in [1], [2, p.251], [10]. These problems represent models of different phenomena arising in chemistry or biology; see [1], [2].

On the other hand, numerical methods for quasilinear singular perturbation problems with $\mu = 1$ were considered in [4]–[6], [8], [12]–[14], just to mention some of the papers.

In this paper our aim is to solve (1.1) numerically by using the approach from [9]–[12], [14], [3]. First, in Section 2, we derive estimates of the derivatives of u_ε. As we may expect, they show boundary layer behaviour of u_ε at $x = 0$ and $x = 1$. (Note that (1.2) guarantees the unique solvability of the reduced problem

$$c(x,u) = 0, \quad x \in I,$$

whose solution, in general, does not satisfy the boundary conditions (1.1b).) Then, in Section 3, we construct a special discretization mesh which is dense in the layers. We form the discrete problem corresponding to (1.5), (1.1b) by using the Lax-Friedrichs finite difference scheme for $p < 1$, and the central scheme for $p \geq 1$. For both schemes we prove uniform (i.e. uniform in ε) stability in the discrete L^1 norm. This norm is used because of the quasilinearity of equation (1.1a) (cf. [4], [5], [8], [12], [13], [14]). In Section 4 we deal with the consistency error using the estimates from Section 2 and properties of the special mesh. As it was shown in [13], the linear uniform convergence of the numerical solution towards the restriction of u_ε on the mesh can be obtained in the discrete L^1 norm even on equidistant meshes. Here, by using the special mesh we are able to improve the L^1 convergence result, cf. [14]. Moreover, numerical results, presented in Section 5, show the pointwise uniform convergence as well. For the case $p \geq 1$ we are able to estimate the maximum error by

$$M(n^{-1} + \varepsilon^{-1}\exp(-m_0 n)),$$

where n is the number of mesh subintervals, and m_0 is a positive constant independent of ε and n. Throughout the paper M will denote any positive constant, independent of ε and n.

§2. Estimates of the Derivatives of u_ε

In this section we shall estimate $|u_\varepsilon^{(k)}(x)|$ for $k = O(1)4, x \in I$. Throughout the section we shall assume (1.2) and that ε^* is sufficiently small. We shall use the technique from [10].

We shall start by giving some rough estimates:

Lemma 2.1.

$$|u_\varepsilon^{(k)}(x)| \leq M\varepsilon^{-k}, \quad k = O(1)4, \quad x \in I.$$

Proof. Because of (1.4) the estimate for $k = 0$ is immediate. Let us now prove the estimate for $k = 1$. If $x \in [0, 1/2]$ we take

$$x^* \in (x, x + \varepsilon) \subset I,$$