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TAYLOR EXPANSION ALGORITHM FOR THE BRANCHING SOLUTION OF THE NAVIER-STOKES EQUATIONS

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Abstract. The aim of this paper is to present a general algorithm for the branching solution of nonlinear operator equations in a Hilbert space, namely the k-order Taylor expansion algorithm, $k \geq 1$. The standard Galerkin method can be viewed as the 1-order Taylor expansion algorithm; while the optimum nonlinear Galerkin method can be viewed as the 2-order Taylor expansion algorithm. The general algorithm is then applied to the study of the numerical approximations for the steady Navier–Stokes equations. Finally, the theoretical analysis and numerical experiments show that, in some situations, the optimum nonlinear Galerkin method provides higher convergence rate than the standard Galerkin method and the nonlinear Galerkin method.

Key Words. Nonlinear operator equation, the Navier-Stokes equations, Taylor expansion algorithm, Optimum nonlinear Galerkin method.

1. Introduction

Many integral equations and differential equations in mathematical physics can be reduced to the operator equations. The operator equations and their numerical approximation are very important in the areas of theoretical mathematics and computational mathematics(see[1]). The main feature of the approximate theory of the operator equations is to apply the functional analytic method to the study of the numerical approximation of the operator equations, which will provide new ideas and new algorithms for the computational mathematics.

This paper is devoted to present the k-order Taylor expansion algorithms for the branching Solution of the nonlinear operator equations. The standard Galerkin (SG) method and the optimum nonlinear Galerkin (ONG) method can be viewed as specific Taylor expansion algorithms.As the important application of the algorithms, we consider the numerical approximations of the 2–D steady Navier-Stokes equations and estimate the convergence rates of the corresponding algorithms.Moreover, we also recall the convergence rate of the nonlinear Galerkin (NG) methods presented in[4–9]. Our theoretical analysis and numerical experiments show that the ONG method is of the higher convergence rate than the NG method and the SG method.

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2. Operator Equation and Taylor Expansion Algorithms

We are given a Hilbert space H with a scalar product$(\cdot,\cdot)$and a norm $|\cdot|$. The abstract operator equation that we will study has the form

$$F(u) = f. \tag{2.1}$$

Here $F : D(F) \subset H \to H$ is a nonlinear operator, $D(F) = v \in H; F(v) =\in H$is the domain of operator F, $f \in H$is given and $u \in D(f)$ is a unknown function(or vector function)defined in a bounded domain Ω of R^2or R^3.

We now recall the following Taylor expansion (see[1]).

Theorem 2.1. *Assume that $F : D(F) \to H$ is the continuous Fréchet differentiable of the k-order. Then for each $p \in D(F)$,$q \in H$, $p+q \in D(F)$ there holds the Taylor expansion with the integral remainder, namely*

$$\begin{aligned} F(p+q) \quad &= F(p) + \tfrac{1}{1!}DF(p)q + \cdots + \tfrac{1}{(k-1)!}D^{k-1}F(p)q^{k-1} \\ &+\tfrac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}D^kF(p+tq)q^k\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

For each $n > 0$, we let H_n be a n–dimensional subspace of H and $P_n : H \to H_n$ be an orthogonal projection operator. To introduce the Taylor expansion algorithm, we select a large n and rewrite the solution u of (2.1) as

$$u = p + q, p = P_n u \in H_n, q = (I - P_n)u \in H\backslash H_n,$$

such that p represents the large eddies of the flow and q represents the small eddies of the flow, namely $|q| \to 0$(as $n \to \infty$) (refer to [2-3]). Hence, we apply respectively P_n and $Q_n = I - P_n$ to (2.1):

$$P_nF(p+q) = P_nf, \tag{2.3}$$

$$Q_nF(p+q) = Q_nf. \tag{2.4}$$

Assume that $F(u) = F(p+q)$ can be rewritten as the Taylor expansion (2.2). Thanks to q being the small eddies of the flow, it is then reasonable to neglect some small terms as $DF(p)q, \frac{1}{2!}DF(p)q^2, \cdots \frac{1}{(k-1)!}\int_0^1(1-t)^{k-1}D^kF(p+tq)q^k\,\mathrm{d}t$, in (2.2). Then we obtain the following Taylor expansion algorithms:
the 1-order Algorithm : Find $u_{app} = y \in H_n$ such that

$$P_nF(y) = P_nf; \tag{2.5}$$

the 2-order Algorithm : find $u_{app} = y + z \in H, y \in H_n, z \in H\backslash H_n$, such that

$$P_nF(y+z) = P_nf, \tag{2.6}$$

$$Q_n(F(y) + DF(y)z) = Q_nf; \tag{2.7}$$

the k-order Algorithm : find $u_{app} = y + z \in H, y \in H_n, z \in H\backslash H_n$, such that

$$P_nF(y+z) = P_nf, \tag{2.8}$$

$$Q_n(F(y) + DF(y)z + \cdots \frac{1}{(k-1)!}D^{k-1}F(y)z^{k-1}) = Q_nf. \tag{2.9}$$

Notice that (2.7),(2.9) are the infinite dimensional system. From the computational point of view we have to replace $H\backslash H_n$ and Q_n by $H_N\backslash H_n$ and $Q_n^N = P_N - P_n$ in (2.7),(2.9), where $N > n$ will be chosen according to some convergence analysis.

In particular, we notice that the 1-order Taylor expansion Algorithm is then the standard Galerkin (SG) method. Moreover, the 2-order Taylor expansion Algorithm