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A METHOD OF FINDING A STRICTLY FEASIBLE SOLUTION FOR LINEAR CONSTRAINTS*

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Abstract

This paper presents a method of finding a strictly feasible solution for linear constraints. We prove, under certain assumptions, that the method is convergent in a finite number of iterations, and give the sufficient and necessary conditions for the infeasiblity of the problem. Actually, it can be considered as a constructive proof for the Farkas lemma.

§1. Introduction

In this paper we consider the following problem: to find a vector $x^{(0)} > 0$ which satisfies the linear constraints

$$Ax = b, \quad x \geq 0 \tag{1.1}$$

where A is an $m \times n$ real matrix with rank m, b is a real vector in R^m, and x is a real variable in R^n. A vector $x^{(0)}$ is called a strictly feasible solution if it satisfies (1.1) and all its components are positive.

This problem arises in solving the standard form of linear programming using an interior point method [7], [8], and minimizing the problem of a nonlinear objective function with linear constraints by means of barrier and penalty functions. Especially, a new polynomial-time algorithm for linear programming [4] was presented in recent years, It is a great improvement on complexity, and furthermore, is said to be 50 times faster than the simplex method for practical problem, Unfortunately, no further information on the test problems or experimental procedures was given. Therefore, it has arosed extensive attention and discussion. The idea of the new algorithm originated from the techniques of solving nonlinear programming problems. Obviously, the essential difference between the new algorithm and the simplex method is that it finds the optimal solution from the interior feasible direction of the constrained region. On the other hand, a similar result can also be deduced from a projected Newton barrier function [2] and the penalty function method [3]. As is well known, these methods all require a strictly feasible starting point for minimization, and generate a sequence of strictly feasible solution. So, how to find an initial strictly feasible solution for problem (1.1) in practice is an important problem.

This paper presents an efficient method of finding a strictly feasible solution for problem (1.1). In fact, the method can be introduced directly from the interior point method, and it leads to computational simplicity. In Section 2 we describe the algorithm, and show how to start it, when to stop it, and how to easily identify infeasibility. In Section 3, under certain assumptions, we prove its convergence to a strictly feasible solution in a finite number of iterations, and the sufficient and necessary conditions for the infeasibility.

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§2. Feasibility and Algorithm

In this section, the necessary and sufficient conditions of the feasibility for problem (1.1) are briefly discussed, and the sufficient conditions of the existence of a strictly feasible solution are given, we give an algorithm for finding a strictly feasible solution in a finite number of iterations or indicating infeasible conditions for problem (1.1) in the case of nondegeneration, because the degeneration case is too complicated to be discussed here.

Concerning the feasibility, actually Farkas's theorem has shown the necessary and sufficient condition of feasibility for problem (1.1).

Lemma 2.1 (Farkas Theorem). *Suppose that $A \in R^{m\times n}, b \in R^m$. Then problem (1.1) is feasible if and only if for all the nonzero vectors $y \in R^m$ which satisfy $A^Ty \geq 0$, the following inequality holds:*

$$b^Ty > 0.$$

Obviously, the existence of the strictly feasible solution is not garanteed when problem (1.1) is feasible, Therefore, it is necessary to have a strong condition in order to ensure the existence of a strictly feasible solution.

Theorem 2.2 *Suppose that $rank(A) = m$, and that problem (1.1) is feasible and nondegenrate. Then there is a strictly feasible solution.*

A constructive proof of the theorem is given in Section 3.

Now we describe our algorithm. Assume that $x^{(0)} > 0$ is a given vector.

Algorithm A:

Let $k = 0$, and let an initial starting point $x^{(0)}$, be given

(1) Define

$$D_k = \mathrm{diag}(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}), \tag{2.1}$$

$$A_k = AD_k \tag{2.2}$$

and compute the residual vector

$$r^{(k)} = b - Ax^{(k)}. \tag{2.3}$$

(2) Compute vector

$$p^{(k)} = A_k^T(A_kA_k^T)^{-1}r^{(k)}. \tag{2.4}$$

If $p^{(k)} \leq 0$, and $b^T(A_kA_k^T)^{-1}r^{(k)} > 0$, stop; then problem (1.1) is infeasible. Otherwise, go to the next step.

(3) Choose the minimum component of $p^{(k)}$, and let

$$P_k = \min_i\{p_i^{(k)}\}. \tag{2.5}$$

If $-1 < P_k$, then

$$x^{(k+1)} = x^{(k)} + D_kp^{(k)}. \tag{2.6}$$

Thus, $x^{(k)}$ is a strictly feasible solution for problem (1.1), stop. Otherwise, go to next step.