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A NONCONFORMING FINITE ELEMENT METHOD OF STREAMLINE DIFFUSION TYPE FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS *

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Abstract

A nonconforming finite element method of streamline diffusion type for solving the stationary and incompressible Navier-Stokes equations is considered. Velocity field and pressure field are approximated by piecewise linear and piecewise constant functions, respectively. The existence of solutions of the discrete problem and the strong convergence of a subsequence of discrete solutions are established. Error estimates are presented for the uniqueness case.

§1. Introduction

Finite element schemes of streamline diffusion type are nowadays a common procedure for solving convection dominated problems in fluid mechanics such as transport problems[5],[9],[1], the Euler equations and Navier-Stokes equations with small viscosity for incompressible[5],[10] or compressible flow[6],[11].

In this note the incompressible stationary Navier-Stokes equations are addressed. Recently, Johnson and Saranen[10] considered streamline diffusion methods for the time-dependent case where discrete velocity fields are employed which are assumed to be exactly divergence free. In the stationary case, we consider a nonconforming FEM based on piecewise linear and piecewise constant approximations for the velocity and the pressure fields, respectively, satisfying the discrete LBB-condition and thus circumventing exact divergence free discrete velocity fields. For other approaches of upwind type concerning the incompressible Navier-Stokes equation, see e.g. [3] or [13].

The plan of the paper is the following. In Section 2 we give some notations. The FEM is presented in Section 3. Existence and uniqueness results for the discrete problems are given in Section 4. Convergence properties of the method are studied in Sections 5 and 6.

§2. Notations

Let $\Omega \subset R^N (N = 2, 3)$ be a convex polygon or polyhedron with boundary $\Gamma = \partial\Omega$ and let $\nu = 1/\mathrm{Re} > 0$. We consider the incompressible Navier-Stokes equation:

Find (u, p) such that

$$\begin{cases} -\nu\Delta u + u\nabla u + \nabla p &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

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The spaces for velocity and pressure are defined by

$$V = H_0^1(\Omega)^N, Q = \left\{q \in L^2(\Omega); \int_\Omega q dx = 0\right\},$$
$$W = \{v \in V; \nabla \cdot v = 0\}.$$

For vector-valued functions $u = (u_1, \cdots, u_N) \in W^{k,p}(\Omega)^N$, $v = (v_1, \cdots, v_N)$ belonging to $L^\infty(\Omega)^N$ we use the usual norms and seminorms, respectively,

$$\|u\|_{k,p}^p = \sum_{i=1}^N \|u_i\|_{k,p}^p, \quad |u|_{k,p}^p = \sum_{i=1}^N |u_i|_{k,p}^p,$$
$$\|v\|_{0,\infty} = \max_i \|v_i\|_{0,\infty}.$$

Furthermore, we introduce

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad \forall u, v \in V,$$
$$b(u,v,w) = \int_\Omega (u \cdot \nabla) v w dx, \qquad \forall u, v, w \in V,$$
$$\bar{b}(u,v,w) = \frac{1}{2}\{b(u,v,w) - b(u,w,v)\}, \quad \forall u, v, w \in V$$

satisfying

$$b(u,v,w) = \bar{b}(u,v,w), \quad \forall u \in W, v, w \in V.$$

The weak velocity-pressure formulation of (2.1) reads now;

$$\begin{cases} \text{Find } (u,p) \in V \times Q \text{ such that} \\ \nu a(u,v) + \bar{b}(u,u,v) - (p, \nabla \cdot v) = (f,v), \quad \forall v \in V, \\ (q, \nabla \cdot u) = 0, \quad \forall q \in Q. \end{cases} \tag{2.2}$$

From (2.2) we obtain the weak velocity formulation

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ \nu a(u,v) + \bar{b}(u,u,v) = (f,v), \quad \forall v \in W. \end{cases} \tag{2.3}$$

It is well-known that (2.2) admits at least one solution and that this solution is unique provided that $\nu^{-2}\|f\|_{0,2}$ is sufficiently small.

§3. A Nonconforming Streamline Diffusion Finite Element Method

We consider a nonconforming finite element approximation due to Crouzeix and Raviart[2], and Temam[14]. Let $(\mathcal{J}_h)_h$ be a regular family of triangulations of Ω into N-simplices K_j