UPPER LIMITATION OF KOLMOGOROV COMPLEXITY AND UNIVERSAL P. MARTIN-LÖF TESTS *

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In this paper we study the Kolmogorov complexity of initial strings in infinite sequences (being inspired by [9]), and we relate it with the theory of P.Martin-Löf random sequences. Working with partial recursive functions instead of recursive functions we can localize on an apriori given recursive set the points where the complexity is small. The relation between Kolmogorov's complexity and P. Martin-Löf's universal tests enables us to show that the randomness theories for finite strings and infinite sequences are not compatible (e.g.because no universal test is sequential).

we lay stress upon the fact that we work within the general framework of a non-necessarily binary alphabet.

Preliminaries

Throughout the paper $\mathbf{N} = \{0, 1, 2, \ldots\}$ will be the set of natural numbers. The integral part of a real number x will be denoted by $[x]$. If A is a finite set, then card A is the number of elements of A.

For every non-empty sets A and B we shall write $f : A \xrightarrow{\circ} B$ to denote a partial function, i.e. a function $f : A' \to B$, where A' is a subset of A. We shall consider that A' is the domain of f and we shall write A'=dom(f). We shall say that f is undefined at x and we shall write $f(x) = \infty$ in case x is not in dom(f). The graph of f is the set $\{(x, f(x)) | x \in \text{dom}(f)\} \subset A \times B$. In case $f, g : A \xrightarrow{\circ} B$ are two partial functions such that dom$(f) \subset$ dom(g) and $g(x) = f(x)$ for every $x \in \text{dom}(f)$, we say that g extends f.

We work with a finite alphabet $X = \{a_1, a_2, \ldots, a_p\}$, where $p \geq 2$ is a fixed natural (the binary case $p = 2$ is the most commonly used). The free monoid generated by X under concatenation is X^*. Its elements are called strings. The length of a string $x = x_1x_2\ldots x_n$ of X^* is $\rho(x) = n$. The empty string λ has length 0. If $x, y \in X^*$, then we write $x \subset y$ in case $y = xz$ for some string z. For every natural $n \geq 1$, put $X^n = \{x \in X^* | \rho(x) = n\}$. The set X^* is lexicographically ordered by $a_1 < a_2 < \ldots < a_p < a_1a_1 < a_1a_2 < \ldots a_1a_p < \ldots$.

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We denote by X^∞ the set of all sequences $x = x_1x_2\dots x_n\dots$, where $x_n \in X$ for all natural $n \geq 1$. For such an $\underline{x}$ and for every natural $n \geq 1$, put $\underline{x}(n) = x_1x_2\dots x_n \in X^*$. If $y \in X^*, y = y_1y_2\dots y_m$ and $\underline{x} \in X^\infty, \underline{x} = x_1x_2\dots x_n\dots$, we shall write $y\underline{x}$ to denote the sequence $y_1y_2\dots y_mx_1x_2\dots x_n\dots$, and $\lambda\underline{x} = \underline{x}$. For every $V \subset X^*$, put $VX^\infty = \{y\underline{x}|y \in V, \underline{x} \in X^\infty\}$. In case V is a singleton, i.e. $V = \{y\}$, we write yX^∞ instead of VX^∞.

For the Recursive Function Theory see [10]. Dealing with computability, we do not distinguish between $\mathbf{N}, \mathbf{N}\backslash\{0\}$ and X^*. A recursively enumerable (r.e.) set is the domain of some partial recursive (p.r.) function.

A P. Martin-Löf test (M-L test in the sequel) is a (possibly empty) r.e. set $V \subset X^* \times (\mathbf{N}\backslash\{0\})$ having the following properties :

i) For every natural $m \geq 1$, one has $V_{m+1} \subset V_m$, where $V_m = \{x \in X^*|(x,m) \in V\}$.

ii) For all natural non-null m and n one has card $(X^n \cap V_m) < p^{n-m}/(p-1)$.

A M-L test having the following additional property :

iii) For every natural $m \geq 1$, and for all strings x, y in X^* with $x \subset y$ and $x \in V_m$ one has $y \in V_m$ is called a sequential M-L test (s. M-L test in the sequel).

A (sequential) M-L test U will be called a universal (universal sequential) M-L test if for every (sequential) M-L test V there exists a natural c (depending upon V and U) such that $V_{m+c} \subset U_m$ for all natural $m \geq 1$. For the existence of universal (universal sequential) M-L tests see [7], [8], [11], [1] and [3].

The critical level induced by a M-L test V is the function $m_V : X^* \to \mathbf{N}$ given by $m_V(x) = \max\{m \in \mathbf{N}|x \in V_m\}$ in case $x \in V_1$, and $m_V(x) = 0$ otherwise.

Let $\varphi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ be a p.r. function. According to A.N. Kolmogorov [6], we define the Kolmogorov complexity $K_\varphi : X^* \times \mathbf{N} \overset{\circ}{\to} \mathbf{N}$ as follows : $K_\varphi(x|n) = \min\{\rho(y)| \ y \in X^*, \varphi(y,n) = x\}$ if such y does exist, and $K_\varphi(x|n) = \infty$ in the opposite case. Now we can define, for every φ as adove, the set $V(\varphi) = \{(x,m) \in X^* \times \mathbf{N}|K_\varphi(x)\rho(x)) < \rho(x) - m\}$. It is readily seen that $V(\varphi)$ is a M-L test (see [1]). A p.r. function $\psi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ is called a universal Kolmogorov algorithm in case for every p.r. function $\varphi : X^* \times \mathbf{N} \overset{\circ}{\to} X^*$ there exists a natural c (depending upon φ and ψ) such that $K_\psi(x|m) \leq K_\varphi(x|m) + c$ for all $x \in X^*$ and $m \in \mathbf{N}$. Universal Kolmogorov algorithms do exist (see [6], [1], [4]). Furthermore, for every universal Kolmogorov algorithm ψ, the M-L test. $V(\psi)$ is universal (see [2] and [4]).

Results

We begin with a slightly improved version of a result in [5].

Lemma 1. *Let $n(1), n(2), \dots, n(k)$ be natural numbers, $k \geq 1$. The following assertions are equivalent :*

A) *One has*

$$\sum_{i=1}^{k} p^{-n(i)} \geq 1. \tag{1}$$