SENSITIVITY ANALYSIS OF MULTIPLE EIGENVALUES (I)*

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Abstract

The technique described in [9] is used to discuss the sensitivity of multiple eigenvalues and corresponding eigenspaces of symmetric eigenproblems analytically dependent on several parameters. The results may be useful for investigating problems of stability and response analysis of linear structures.

§ 1. Introduction

Throughout this paper we use the following notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$, $\mathbb{R}=\mathbb{R}^1$ and

$$\mathbb{R}_r^{m\times n}=\{A\in\mathbb{R}^{m\times n}: \operatorname{rank}(A)=r\},\ S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}: A^T=A\},$$

in which the superscript T is for transpose. $I^{(n)}$ is the $n\times n$ identity matrix, and 0 is the null matrix. $A>0$ denotes that A is a positive definite matrix. We use $\rho(\ )$ for the spectral radius and $\|\ \|_2$ the usual Euclidean vector norm. The set of the eigenvalues of an eigenproblem $Ax=\lambda x$ is denoted by $\lambda(A)$ and the set of the eigenvalues of the eigenproblem $Ax=\lambda Bx$ is denoted by $\lambda(A, B)$. Besides, let $\lambda_1(A), \cdots, \lambda_n(A)$ denote an eigenvalues of an $n\times n$ matrix A.

Let $p=(p_1, p_2, \cdots, p_N)^T\in\mathbb{R}^N$. Suppose that $A(p)=(\alpha_{ij}(p))$, $B(p)=(\beta_{ij}(p))\in S\mathbb{R}^{n\times n}$ are real analytic functions in some neighbourhood $\mathscr{B}(p^*)$ of the point $p^*\in\mathbb{R}^N$ and $B(p)>0\ \forall p\in\mathscr{B}(p^*)$. Without loss of generality we may assume that the point p^* is the origin of $\mathbb{R}^N$. It is well known that the eigenproblem

$$A(p)x(p)=\lambda(p)B(p)x(p),\quad \lambda(p)\in\mathbb{R},\ x(p)\in\mathbb{R}^n,\ p\in\mathscr{B}(0) \tag{1.1}$$

arises frequently in structural design, and it is often desirable to be able to estimate the sensitivity of the available designs to changes in system parameters. Although investigation of the sensitivity of eigenvalues and eigenvectors has a long history[3,4,9], the case of multiple eigenvalues is rarely treated in the literature (see [8, p. 606], [6, p. 133], [9, p. 362], [5]). The object of this paper is to discuss the sensitivity of multiple eigenvalues and corresponding eigenspaces of the eigenproblem (1.1) with respect to the parameters $p_1, p_2, \cdots, p_N$. The results of this paper may be useful for investigating problems of stability and response analysis of linear structures.

The following example is very interesting, from which we shall gain a good deal of enlightenment.

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Example 1.1[1, p.386], [9, p.362]. We consider an eigenproblem

$$A(p)x(p)=\lambda(p)x(p) \tag{1.2}$$

with

$$A(p)=\begin{pmatrix} 1+2p_1+2p_2 & p_2 \\ p_2 & 1+2p_2 \end{pmatrix},\ \lambda(p)\in\mathbb{R},\ x(p)\in\mathbb{R}^2,\ p=(p_1, p_2)^T\in\mathbb{R}^2. \tag{1.3}$$

Obviously, the matrix $A(p)$ is a real analytic function of $p\in\mathbb{R}^2$, $A(0)$ has eigenvalue 1 with multiplicity 2 and the eigenvalues of $A(p)$ are

$$\lambda_1(p)=1+p_1+2p_2+\sqrt{p_1^2+p_2^2},\quad \lambda_2(p)=1+p_1+2p_2-\sqrt{p_1^2+p_2^2}. \tag{1.4}$$

It is easy to see the following facts:

$$\begin{aligned} &\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\,p_1=+0}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\,p_1=-0}=2, \\ &\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\,p_1=-0}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\,p_1=+0}=0, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\,p_2=+0}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\,p_2=-0}=3, \\ &\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\,p_2=-0}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\,p_2=+0}=1; \end{aligned} \tag{1.6}$$

$$\lambda\left(\left(\frac{\partial A(p)}{\partial p_1}\right)_{p=0}\right)=\{2,\ 0\},\quad \lambda\left(\left(\frac{\partial A(p)}{\partial p_2}\right)_{p=0}\right)=\{3,\ 1\}. \tag{1.7}$$

Here we define

$$\left(\frac{\partial\lambda_s(p)}{\partial p_1}\right)_{p=0,\,p_1=+0}=\lim_{p_1\to+0}\frac{\lambda_s(p_1,\ 0)-\lambda_s(0,\ 0)}{p_1}$$

and

$$\left(\frac{\partial\lambda_s(p)}{\partial p_1}\right)_{p=0,\,p_1=-0}=\lim_{p_1\to-0}\frac{\lambda_s(p_1,\ 0)-\lambda_s(0,\ 0)}{p_1},\quad s=1,\ 2.$$

The partial derivatives $\left(\frac{\partial\lambda_s(p)}{\partial p_2}\right)_{p=0,\,p_2=+0}$ and $\left(\frac{\partial\lambda_s(p)}{\partial p_2}\right)_{p=0,\,p_2=-0}$ $(s=1,\ 2)$ are defined similarly.

The relations (1.4)—(1.7) show that the functions $\lambda_1(p)$ and $\lambda_2(p)$ are not differentiable (thus they are not analytic) at $p=0$, but there exist two permutations π and π' of 1, 2 such that one has the relations

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\,p_i=+0}=\lambda_{\pi(s)}\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}\right) \tag{1.8}$$

and

$$\left(\frac{\partial\lambda_s(p)}{\partial p_i}\right)_{p=0,\,p_i=-0}=\lambda_{\pi'(s)}\left(\left(\frac{\partial A(p)}{\partial p_i}\right)_{p=0}\right),\quad s=1,\ 2,\ i=1,\ 2. \tag{1.9}$$

In the next section we shall prove that the relations (1.8) and (1.9) are of universal significance, on the basis of which we may define the sensitivity of multiple eigenvalues dependent on several parameters, and give some formulae for computing the sensitivity (see § 3).

Remark 1.1. Let $A(p)$ be the matrix described in (1.3). If we set

$$\hat{A}(p_1)=(A(p))_{p=(p_1,0)^T},\quad \tilde{A}(p_2)=(A(p))_{p=(0,p_2)^T},$$

then $\hat{A}(0)$ and $\tilde{A}(0)$ have eigenvalue 1 with multiplicity 2, the eigenvalues of $\hat{A}(p_1)$ are