NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS WITH SINGULAR PERTURBATION CONSIDERATIONS*[1)]

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Abstract

A numerical method for coarse grids is proposed for the numerical solution of the incompressible Navier–Stokes equations. From singular perturbation considerations, we obtain partial differential equations and boundary conditions for the outer solution and the boundary layer correction. The former problem is solved with the finite difference method and the latter with the approximate method. Numerical experiments show that accurate outer flow and boundary flux result with little computational effort.

§ 1. Introduction

It is well known that partial differential equations exhibit multiple scale phenomena. To resolve all the different scales in numerical computation presents quite a challenge to the computational fluid dynamic community, see Chin, Hedstrom, and Howes [3]. Even for the numerical solution of boundary layer problems, affordable uniform grids are often too coarse to resolve such layers. Usually fine grids are used in the boundary layers for small scale effects represented by large gradients in the solutions, while coarse grids are used in most of the region for large scale phenomena described by smooth solutions, see e.g. MacCormack and Lomax [14]. Or, the viscous and inviscid equations can be coupled and solved iteratively, see e.g. Van Dalsem and Steger [19]. Both approaches need great computational effort. Often for practical purposes detailed resolution of the boundary layers is not necessary; only the boundary fluxes (shearing stress, heat flux, etc.) in terms of normal derivatives at the boundaries are needed. This will allow the use of more efficient numerical schemes.

The present paper is concerned with the numerical solution of the incompressible Navier–Stokes equations in primitive variables with large Reynolds numbers. It presents a method with singular perturbation considerations; concepts of outer solution, boundary layer correction, etc. are taken over, but not the solution procedures which consists of finding the outer and inner solutions of various orders successively. Our method is, first of all, to decompose the solution into two parts, an outer part which is smooth and a boundary layer correction part with large gradients, coupled essentially through suitable boundary conditions at the fixed boundary. Thus different methods can be applied to the two different problems,

* Received January 5, 1987.
1) The Project Supported by the National Natural Science Fundation of China.

giving numerical solution to the complete Navier-Stokes equations if necessary.

In this paper, for smooth flow in the main part of the computational domain, a finite difference methods on a coarse grid is used for its numerical solution. At the boundaries where the solution has large gradients, if the analytic behavior is known, then efficient numerical schemes can be developed, as the exponential schemes for two-point boundary value problems, see e.g. Doolan, Miller and Schilders [5]. For two-dimensional scalar partial differential equations, the analytic behavior of the solution near characteristic boundaries has been investigated by Eckhaus and de Jager [7], Howes [11], etc. But for our nonlinear system of partial differential equations, to the authors' knowledge, no simple analytic function can capture the essential behavior of the solution in the boundary layers. Hence approximate methods are applied directly to a simplified system of equations for the boundary layers. This system is to be formulated in terms of boundary layer correction, since the authors find it mathematically more tractable than that of outer and inner solutions with matched asymptotic expansion. However, the class of approximate methods for Prandtl's boundary layer equations due to von Karman and Pohlhausen, see Schlichting [17, Chap. 10], can be adapted to the numerical solution of the boundary layer correction equations. In this sense, our method is similar to the coupled inviscid integral-boundary-layer algorithm quoted in [19]. But here the complete outer solution can be obtained, not just the first order inviscid solution; and then corrections are added at the boundaries, with no matching because of our formulation. Numerical solution with boundary layer correction is also given in Flaherty and O'Mallay [8] for one dimensional problems, but with semi-analytic considerations and a different solution procedure.

In the following, Section 2 presents the basic idea of our method with a simple linear scalar differential equation. Section 3 focuses on the incompressible Navier-Stokes equations. The outer system of equations and its boundary conditions are discussed, and the boundary layer correction equations and the corresponding boundary conditions are derived. Then in Section 4 the method of solution is presented, and finally in Section 5 test results of our method on incompressible flow past a semi-infinite plate are given. These preliminary numerical experiments prove the feasibility of our present approach.

§ 2. The Basic Idea

We present the basic idea of our numerical method with a simple example from O'Malley [15, Chap. 1]——a linear scalar differential equation with constant coefficients

$$\varepsilon \frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0 \tag{2.1}$$

for $0\leqslant x\leqslant 1$, with boundary conditions

$$y(0)=0, \tag{2.2}$$

$$y(1)=1. \tag{2.3}$$

Its exact solution is