# QUADRATURE FORMULAS FOR SINGULAR INTEGRALS WITH HILBERT KERNEL*1)

DU JIN-YUAN (杜金元)
(*Wuhan University, Wuchang, China*)

**Abstract**

In this paper, we first establish the quadrature formulae of proper integrals with weight by trigonometric interpolation. Then we use the method of separation of singularity to derive the quadrature formulae of corresponding singular integrals with Hilbert kernel. The trigonometric precision, the estimate of the remainder and the convergence of each quadrature formula derived here are also established.

## § 1. Introduction

We shall consider the numerical evaluation of singular integrals with Hilbert kernel of the form

$$I(f, x)=\int_0^{2\pi} w(t)f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt, \quad x\in[0, 2\pi), \tag{1.1}$$

where $w(t)$ is a given non-negative function with period $2\pi$ which is known as the weight function, and $f(t)$ is a function with period $2\pi$. $w(t)$ and $f(t)$ are assumed further to be Hölder-continuous for the existence of (1.1)[1].

The investigations on numerical evaluation of singular integrals with Cauchy kernel are rather complete[2-6]. But the results of investigations on numerical evaluation for singular integrals with Hilbert kernel are not many up to now, except for some special cases.

In 1974, M. M. Chawla and T. R. Ramakrishnan discussed the numerical evaluation of (1.1) for the case $w(t)=1$. They assumed that $f(z)$ is a $2\pi$-periodic function analytic on the rectangular domain $D_r=\{z,\ 0\leqslant \operatorname{Re}(z)\leqslant 2\pi,\ -r\leqslant \operatorname{Im}(z)\leqslant r,\ r>0\}$ with the boundary $B_r$, which is written as $f\in AP(B_r)$. For such functions, evaluating directly the contour integral

$$(4\pi i)^{-1}\int_{B_r}\left[f(z)\operatorname{ctg}\frac{1}{2}(z-x)/\sin(nz)\right]\operatorname{ctg}\frac{1}{2}(z-t)dz$$

by the analyticity of $f(z)$, they obtained the following quadrature formula[7]:

$$\int_0^{2\pi} f(t)\operatorname{ctg}\frac{1}{2}(t-x)dt=Q_n(f, x)+R_n(f, x), \tag{1.2}$$

where

* Received April 29, 1986.
1) Projects supported by the Science Fund of the Chinese Academy of Sciences.

$$Q_n(f,\ x)=\begin{cases}\dfrac{\pi}{n}\sum\limits_{k=0}^{2n-1} f(t_k)\operatorname{ctg}\dfrac{1}{2}(t_k-x)+2\pi f(x)\operatorname{ctg}(nx), & x\neq t_k,\ k=0,\ 1,\ \cdots,2n-1,\\ \dfrac{\pi}{n}\sum\limits_{k=0}^{2n-1}{}' f(t_k)\operatorname{ctg}\dfrac{1}{2}(t_k-x)+2\pi f'(t_s)/n, & x=t_s,\ 0\leqslant s\leqslant 2n-1,\end{cases}$$

$$R_n(f,\ x)=(4\pi i)^{-1}\int_{B_r}\left[f(z)\operatorname{ctg}\frac{1}{2}(z-x)S_n(z)/\sin(nz)\right]dz,$$

where $t_k=k\pi/n$,

$$S_n(z)=\int_0^{2\pi}\sin(nt)\operatorname{ctg}\frac{1}{2}(z-t)\,dt,$$

and $\sum'$ denotes the summation for $k$ except $k=s$.

In 1983, N. I. Ioakimidis rediscussed the quadrature formula (1.2) for the purpose of numerical solution of the singular integral equation with Hilbert kernel

$$ay(t)+\frac{b}{2\pi}\int_0^{2\pi}y(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt+\int_0^{2\pi}k(t,\ x)y(t)dt=f(x),\quad 0\leqslant x<2\pi \tag{1.3}$$

with constant coefficients $a$ and $b$. He extended the result of Chawla and Ramakrishnan to the case that the number of nodes may also be odd. Only assuming $f\in C'_{2\pi}$, he obtained[8]

$$\int_0^{2\pi}f(t)\operatorname{ctg}\frac{1}{2}(t-x)\,dt\approx\frac{2\pi}{n}\sum_{k=0}^{n-1}f(t_k)\operatorname{ctg}\frac{1}{2}(t_k-x)+2\pi\operatorname{ctg}\frac{1}{2}nxf(x),$$
$$x\neq t_k,\ k=0,\ 1,\ \cdots,n-1, \tag{1.4}$$

where $t_k=2k\pi/n$.

Ioakimidis pointed out that the quadrature formula (1.4) is exact when $f(t)=\sin(jt)$ $(j=-n+1,\ -n+2,\ \cdots,\ n-1)$, $f(t)=\cos(jt)$ $(j=-n,\ -n+1,\ \cdots,\ n)$. But he gave neither the estimate of the remainder of (1.4) nor the convergence of (1.4). Ioakimidis' method applied to the derivation of (1.4) is the method of separation of singularity. By this method, (1.4) is converted directly to the classical quadrature formula for periodic functions.

The investigations of the numerical evaluation of (1.1) in the case that $w(t)$ is a general weight function, to the author's knowledge, have not appeared in literature until now. It is very natural to consider the weight function in the investigation of the numerical evaluation of singular integrals with Cauchy kernel. In general, the weight function possesses the weak singularity at the end-points of the interval of integration, and hence separates this singularity from the integrand. Particularly, this separation is one of the foundations of the numerical method of singular integral equations with Cauchy kernel[9-11]. In the singular integral with Hilbert kernel (1.1) we require the weight $w(t)$ be Hölder continuous. Hence its separation from the function $f(t)$ is often thought unnecessary, but recently the author found that the theory of numerical evaluation of the singular integral (1.1) with weight $w(t)$ is very significant in practice. In concrete terms, the quadrature formula (1.4) can only apply to the numerical solution of the singular integral equation (1.3) with the constant coefficients, which is discussed by Ioakimidis. For the numerical method of general singular integral equations with Hilbert kernel, it is a material matter to separate the weight $w(t)$ from the function $f(t)$. Thus, the quadrature formula of (1.1) is essential (this problem will be discussed in another