SIMPLICIAL METHODS AND APPROXIMATION OF SEVERAL SOLUTIONS*

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Since Scarf gave in [1] his constructive proof of Brouwer's fixed-point theorem, simplicial methods for solving nonlinear equations have developed rapidly. In 1972, Eaves[2] and Merrill[3] made a substantial improvement on Scarf's algorithm. They approximated zeros of a continuous mapping f by solving a series of piecewise linear approximations of $f(x)=0$. Their work made simplicial methods practical and efficient. Simplicial methods and homotopy continuation methods are closely related. But the former only require that the involved mapping is continuous and need no calculation of derivatives. As a constructive implementation of degree arguments, simplicial methods have a wider range of application than classical iterative methods which are based on the contraction mapping principle. For the survey of simplicial methods and their application, see [4].

There are a lot of discussions on the properties of simplicial methods. It is known that simplicial methods are closely related to the Brouwer degree theory. In this paper we intend to use the properties of piecewise linear approximations of continuous mappings and the Brouwer degree theory to analyze Eaves and Saigal's deformation method (see [2], [6]) and apply our results in discussing the problem of approximating several solutions.

§ 1. Definitions, Notations and Preliminaries

Let T be a triangulation of R^n (for definition, see [4]). For $0\leqslant m\leqslant n$ we denote by T^m the collection of all m-faces (i.e. convex closures of $m+1$ vertices of a simplex in T). Especially $T^n=T$; T^0 is composed of all vertices of elements of T. Define

$$|T^m|=\bigcup_{\sigma\in T^m}\sigma,\quad 0\leqslant m\leqslant n,$$

$$\partial T=\{\tau\in T^{n-1};\ \text{there is only one } \sigma\in T \text{ such that } \tau\subset\sigma\},$$

$$|\partial T|\bigcup_{\tau\in\partial T}\tau.$$

For $\sigma\in T$, define

$$\operatorname{diam}(\sigma)=\max_{x,y\in\sigma}\|x-y\|,$$

$$\theta(\sigma)=\rho/\operatorname{diam}(\sigma),$$

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where $\|\cdot\|$ is a given norm and ρ is the radius of the maximum sphere contained in σ. Define

$$\operatorname{mesh}(T)=\sup_{\sigma\in T}\operatorname{diam}(\sigma),$$

$$\theta(T)=\inf_{\sigma\in T}\theta(\sigma).$$

For $\varepsilon>0$ we define $\varepsilon T=\{\varepsilon\sigma;\ \sigma\in T\}$. It is obvious that $\operatorname{mesh}(\varepsilon T)=\varepsilon\operatorname{mesh}(T)$ and $\theta(\varepsilon T)=\theta(T)$.

We call continuous mapping $f\colon |T|\to R^n$ a labeling. There exists a unique f_T: $|T|\to R^n$ such that

(1) $f_T(x)=f(x)$, if $x\in T^0$,

(2) f_T is affine on every $\sigma\in T$.

Hence for every $\sigma\in T$, there are a unique $n\times n$ matrix A_σ and a unique $b_\sigma\in R$ such that $f_T(x)=A_\sigma x+b_\sigma$ for $x\in\sigma$.

Let $\sigma\in T$. We say that σ is f-completely labeled if there is an $\varepsilon_0>0$ such that $(\varepsilon, \varepsilon^2, \cdots, \varepsilon^n)^T\in f_T(\sigma)$ if $0\leqslant\varepsilon\leqslant\varepsilon_0$.

The following lemma is easy to prove by a compact argument.

Lemma 1. *Let $f\colon R^n\to R^n$ be a continuous mapping and $C\subset R^n$ a bounded closed set such that $0\bar{\in} f(C)$. Then there exists an $\eta_0>0$ such that if $\operatorname{mesh}(T)\leqslant\eta_0$, f_T has no zero-point on any $\sigma\in T$ if $\sigma\cap C\neq\emptyset$.*

Let T be a triangulation of R^n and $f\colon R^n\to R^n$ a continuous mapping. Define mapping $I_f\colon T\to\{-1, 1, 0\}$ by

$$I_f(\sigma)=\begin{cases}\operatorname{sign}\det A_\sigma, & \text{if } \sigma \text{ is } f\text{-completely labeled,}\\ 0, & \text{otherwise.}\end{cases}$$

Now we consider triangulations of $R^n\times[a, b]$. If T is such a triangulation, it is obvious that $|\partial T|=R^n\times\{a\}\cup R^n\times\{b\}$. We call a mapping $\lambda\colon R^n\times[a, b]\to R^n$ a labeling on $[T]$, and define λ_t by

$$\lambda_t(x)=\lambda(x, t),\quad x\in R^n,\ t\in[a, b].$$

If $\tau\in T^n$ and there exists an $\varepsilon_0>0$, such that $(\varepsilon, \varepsilon^2, \cdots, \varepsilon^n)\in\lambda_T(\tau)$ if $0\leqslant\varepsilon\leqslant\varepsilon_0$, we call τ a λ-completely labeled n-face. For a given $\sigma\in T$, σ contains either two or no λ-completely labeled n-faces (see [4]). We call $\sigma\in T$ an almost λ-completely labeled $n+1$-simplex if it has two λ-completely labeled n-faces.

Let a λ-completely labeled n-face τ_0 be given. Then τ_0 decides a unique chain of λ-completely labeled n-foces $\{\tau_i\}_{i=N_1}^{N_2}$, where $-\infty\leqslant N_1\leqslant 0\leqslant N_2\leqslant+\infty$ and $N_1\neq N_2$. If both N_1 and N_2 are finite, we say that this λ-completely labeled chain is finite. If N_1 (resp. N_2) is finite, we say that τ_{N_1} (resp. τ_{N_2}) is an end face of $\{\tau_i\}_{i=N_1}^{N_2}$.

Lemma 2. *Let T be a triangulation of $R^n\times[a, b]$, let $\lambda\colon |T|\to R^n$ be given and assume that τ_{N_1}, τ_{N_2} are end faces of a finite λ-completely labeled chain $\{\tau_i\}_{i=N_1}^{N_2}$ in T. If both τ_{N_1} and τ_{N_2} are in $R^n\times\{a\}$ (resp. $R^n\times\{b\}$), we have*

$$I_{\lambda_a}(\tau_{N_1})=-I_{\lambda_a}(\tau_{N_2})\ (\text{resp. } I_{\lambda_b}(\tau_{N_1})=-I_{\lambda_b}(\tau_{N_2})).$$

If $\tau_{N_1}\subset R^n\times\{a\}$ and $\tau_{N_2}\subset R^n\times\{b\}$, we have

$$I_{\lambda_a}(\tau_{N_1})=I_{\lambda_b}(\tau_{N_2}).$$

Lemma 3. *Let $f\colon R^n\to R^n$ be continuous and $\Omega\subset R^n$ a bounded open set such that $0\bar{\in} f(\partial\Omega)$. Then there is an $\eta_0>0$ such that*