A DIFFERENCE SCHEME FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS OF HIGH ORDER*

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§ 1. Introduction

It is well known that the nonlinear equations of Schrödinger type are of great importance to physics and can be used to describe extensive physical phenomena[1,2]. Many authors have discussed the equations of Schrödinger type theoretically and a lot of numerical methods have been presented[3-6].

In this paper, we will consider a class of system of nonlinear Schrödinger equations of high order

$$i\frac{\partial \boldsymbol{u}}{\partial t}+(-1)^m\frac{\partial^m}{\partial x^m}\left(A(x)\frac{\partial^m \boldsymbol{u}}{\partial x^m}\right)+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+F(x,t)\boldsymbol{u}=0, \tag{1.1}$$

with the initial condition

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x),\quad 0\leqslant x\leqslant D, \tag{1.2}$$

and the homogeneous boundary conditions

$$\left.\frac{\partial^l \boldsymbol{u}}{\partial x^l}\right|_{x=0}=\left.\frac{\partial^l \boldsymbol{u}}{\partial x^l}\right|_{x=D}=0,\quad l=0,\cdots,m-1,\ t\geqslant 0, \tag{1.3}$$

where $i=\sqrt{-1}$, $\boldsymbol{u}=(u_1(x,t),\cdots,u_M(x,t))$ is an unknown M-dimensional vector function, $|\boldsymbol{u}|^2=|u_1|^2+\cdots+|u_M|^2$. Both $F(x,t)=(f_{l,r}(x,t))_{M\times M}$ and $A(x)=\operatorname{diag}(a_1(x),\cdots,a_M(x))$ are given real function matrices which are symmetric, $\beta(x)$ and $q(x)$ are given real functions, and $\boldsymbol{u}_0(x)$ is a given M-dimensional complex vector function satisfying condition (1.3).

Corresponding to the problem (1.1)—(1.3), we present a class of difference schemes which satisfy some important conservation laws of equations (1.1). The convergence and stability of the proposed scheme is derived.

§ 2. Establishment of the Difference Scheme

First we introduce some notations. Let $\Omega=[0,D]$, $Q_T=\Omega\times[0,T]$ be a rectangular region. We divide the domain Q_T into small grids by the parallel lines $x=x_j=jh$; $t=t_n=nk$ ($j=0,\cdots,J$; $n=0,\cdots,N$) where $Jh=D$, $Nk=T$. Let $Q_h=\{(x,t);\ x=jh,\ t=nk,\ j=0,\cdots,J;\ n=0,\cdots,N)\}$, and let ϕ_j^n ($j=0,\cdots,J$; $n=0,\cdots,N$) denote the vector valued discrete function on the grid point (x_j,t_n).

Define

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$$\Delta_+ V_j = V_{j+1} - V_j,\ \Delta_- V_j = V_j - V_{j-1},\ f_{\bar{t}}(x, t) = [f(x, t) - f(x, t-k)]/k.$$

We also introduce the inner product and norms appropriate to functions defined on the lattice Q_h. Suppose $f = (f_1, \cdots, f_M)^T$, $g = (g_1, \cdots, g_M)^T$. Then

$$(f, g)_h = \sum_{j=m}^{J-m} \langle f(x_j), g(x_j)\rangle h,$$

$$\|f\|_h^2 = (f, f)_h, \quad \|f\|_\infty = \sup_{m<j<J-m} |f(x_j)|,$$

where $\langle f(x), g(x)\rangle = \sum_{i=1}^{M} f_i(x)\bar{g}_i(x)$, $|f|^2 = \langle f, f\rangle$. The norms corresponding to the space of square integrable functions are

$$\|f\|^2_{L^2(\Omega)} = \int_0^D |f(x)|^2 dx, \quad \|f\|_{L^\infty(\Omega)} = \operatorname*{esssup}_{x\in\Omega} |f(x)|.$$

Corresponding to (1.1)—(1.3), we construct following difference scheme

$$\begin{cases} i\dfrac{\phi_j^{n+1} - \phi_j^n}{k} + (-1)^m \dfrac{\Delta_+^m (A_{j-\frac{m}{2}} \Delta_-^m \phi_j^{n+\frac{1}{2}})}{h^{2m}} + \beta_j P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)\phi_j^{n+\frac{1}{2}} \\ \quad + F_j^{n+\frac{1}{2}} \cdot \phi_j^{n+\frac{1}{2}} = 0, \quad j = m, \cdots, J-m, & (2.1) \\ \phi_j^0 = \phi_{hj}, \quad j = 0, \cdots, J, & (2.2) \\ \Delta_+^l \phi_0^n = \Delta_-^l \phi_J^n = 0, \quad 0 \leqslant n \leqslant N;\ l = 0, \cdots, m-1, & (2.3) \end{cases}$$

where $\phi_j^{n+\frac{1}{2}} = \frac{1}{2}(\phi_j^{n+1} + \phi_j^n)$, $F_j^{n+\frac{1}{2}} = F\left(x_j, t_n + \frac{1}{2}k\right)$, $P(u, v) = (Q(u) - Q(v))/(u - v)$, $u \neq v$; $P(u, u) = q(u)$, $Q(z) = \int_0^z q(s)ds$, $\boldsymbol{\phi}_{hj} = \boldsymbol{u}_0(x_j)$ $(j = m, \cdots, J-m)$ and $\boldsymbol{\phi}_{hj} = \mathbf{0}$ $(j = 0, \cdots, m-1;\ J-m+1, \cdots, J)$. For convenience, we will replace $\boldsymbol{\phi}_j^n$ with ϕ_j^n, $\boldsymbol{u}(x, t)$ with $u(x, t)$ and so on.

By the method in [7], we can get

Theorem 2.1. *The solution* ϕ_j^n $(j = 1, \cdots, J;\ n = 1, \cdots, N)$ *of the difference problem* (2.1)—(2.3) *exists, and is unique if* $q(r) \in C^1[0, \infty)$ *and* k *is small enough.*

§ 3. Priori Estimations for Difference Solution

In this section, we will get a series of priori estimates for the solution of difference equation (2.1)—(2.3).

Lemma 3.0. *For any* $\{u_j\}$ *and* $\{v_j\}$ $(j = 0, \cdots, J)$ *there is a relation*

$$\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j - u_0 v_0 + u_J v_J.$$

Lemma 3.1. *Suppose* $u_0(x) \in C(\Omega)$. *Then there exists* h_0, *such that*

$$\|\phi_j^n\|_h^2 = \|\phi_j^0\|_h^2 \leqslant 2\int_0^D |u_0(x)|^2 dx = E_0, \quad n = 0, \cdots, N;\ 0 < h \leqslant h_0. \tag{3.1}$$

The first equality of (3.1) indicates that the solution of the difference problem (2.1)—(2.3) is conservative of energy like the original problem.

Lemma 3.2. *Suppose the following conditions are satisfied*

(1) $\beta^* \geqslant \beta(x) \geqslant 0$, *for* $x \in [0, D]$,

(2) *for any* $s \in [0, \infty)$, $q(s) \geqslant 0$, $q'(s) \geqslant 0$, $q(s) \in C^1$,

(3) *for any* $1 \leqslant l \leqslant M$, $0 \leqslant x \leqslant D$, $0 < a_* \leqslant a_l(x) \leqslant a^*$,