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CANONICAL INTEGRAL EQUATIONS OF STOKES PROBLEM*

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Abstract

The canonical boundary reduction, suggested by Feng Kang, also can be applied to the bidimensional steady Stokes problem. In this paper we first give the representation formula for the solution of the Stokes problem via two complex variable functions. Then by means of complex analysis and the Fourier analysis, we find the expressions of the Poisson integral formulas and the canonical integral equations in three typical domains. From these results the canonical boundary element method for solving the Stokes problem can be developed.

§ 1. Introduction

Since Feng Kang suggested the canonical boundary reduction[1], which reduces a boundary value problem of an elliptic equation to a singular integral equation on the boundary via Green's formula and Green's function, this method already has been applied to the harmonic boundary value problem, the biharmonic boundary value problem and the plane elasticity problem[2,3,6,8]. This kind of reduction conserves the essential characteristics of the original boundary value problem and occupies a particular place in all boundary reductions. From this, a new numerical method, i.e. the canonical boundary element method, has been developed[2,4,6,7]. It also has many distinctive advantages and is fully compatible with the classical FEM[5-7].

Now we consider the steady Stokes problem, which represents the steady flow of an incompressible viscous fluid with a small Reynolds number. We will study the canonical boundary reduction of the bidimensional steady Stokes problem, and find its Poisson formulas and its canonical integral equations in three typical domains.

§ 2. The Principle of the Canonical Boundary Reduction

Consider the boundary value problems in a plane domain Ω with a smooth boundary Γ:

$$\begin{cases} -\nu\Delta \boldsymbol{u}+\operatorname{grad} p=0, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0, & \\ \boldsymbol{u}=\boldsymbol{u}_0, & \text{on } \Gamma \end{cases} \tag{1}$$

and

$$\begin{cases} -\nu\Delta \boldsymbol{u}+\operatorname{grad} p=0, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0, & \\ \sum_{j=1}^{2}\sigma_{ij}(\boldsymbol{u}, p)n_j=g_i, \quad i=1, 2, & \text{on } \Gamma, \end{cases} \tag{2}$$

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where the coefficient $\nu>0$ is the dynamic viscosity of the flow, the unknowns are the velocity $\boldsymbol{u}$ of the fluid occupying Ω and its pressure p,

$$(\boldsymbol{u}, p)\in (H^1(\Omega))^2\times L^2(\Omega)/\mathbb{R} \quad \text{for } \Omega \text{ bounded},$$

or

$$(\boldsymbol{u}, p)\in (W_0^1(\Omega))^2\times L^2(\Omega) \qquad \text{for } \Omega \text{ unbounded},$$

where

$$W_0^1(\Omega)=\left\{\frac{u}{\sqrt{1+r^2}\ln(2+r^2)}\in L^2(\Omega),\ \frac{\partial u}{\partial x_i}\in L^2(\Omega),\ \ i=1,\ 2,\ r=\sqrt{x_1^2+x_2^2}\right\},$$

$$\varepsilon_{ij}(\boldsymbol{u})=\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\Big/2,$$

$$\sigma_{ij}(\boldsymbol{u}, p)=-\delta_{ij}p+2\nu\varepsilon_{ij}(\boldsymbol{u}),\quad i,\ j=1.\ 2,$$

$\boldsymbol{n}=(n_1, n_2)^T$ is the unit outward normal to Γ. We know that Green's formula for the steady Stokes problem is[9]

$$\begin{aligned}2\nu \sum_{i,j=1}^{2}\int_\Omega \varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})dx&\\ +\sum_{i=1}^{2}\int_\Omega\left(\nu\Delta u_i-\frac{\partial p}{\partial x_i}\right)v_i\,dx-\sum_{i=1}^{2}\int_\Omega p\,\frac{\partial v_i}{\partial x_i}\,dx&\\ =\sum_{i,j=1}^{2}\int_\Gamma \sigma_{ij}(\boldsymbol{u}, p)n_jv_i\,ds.&\end{aligned}\tag{3}$$

From this we can obtain the second Green's formula

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega\left[\left(\nu\Delta u_i-\frac{\partial p}{\partial x_i}\right)v_i-\left(\nu\Delta v_i-\frac{\partial q}{\partial x_i}\right)u_i\right]dx-\sum_{i=1}^{2}\int_\Omega\left(p\,\frac{\partial v_i}{\partial x_i}-q\,\frac{\partial u_i}{\partial x_i}\right)dx\\ &=\sum_{i,j=1}^{2}\int_\Gamma[\sigma_{ij}(\boldsymbol{u}, p)n_jv_i-\sigma_{ij}(\boldsymbol{v}, q)n_ju_i]ds.\end{aligned}\tag{4}$$

Now by putting in (4) $(\boldsymbol{u}, p)=(\boldsymbol{u}(x), p(x))$, the solutions of the Stokes problem, and $(\boldsymbol{v}, q)=(\boldsymbol{G}_1(x, x'), Q_1(x, x'))$ or $(\boldsymbol{G}_2(x, x'), Q_2(x, x'))$ respectively, where $\boldsymbol{G}_1=(G_{11}, G_{12})$, $\boldsymbol{G}_2=(G_{21}, G_{22})$, $G_{ij}(x, x')$ and $Q_i(x, x')$, $i, j=1, 2$, are Green's functions of the Stokes problem in domain Ω, which satisfy

$$\begin{cases}-\nu\Delta G_{ij}(x, x')+\dfrac{\partial}{\partial x_j}Q_i(x, x')=\delta_{ij}\delta(x-x'), & i,\ j=1,\ 2,\\ \displaystyle\sum_{j=1}^{2}\frac{\partial}{\partial x_j}G_{ij}(x, x')=0, & i=1,\ 2,\\ G_{ij}(x, x')_{x\in\Gamma}=0, & i,\ j=1,\ 2,\end{cases}\tag{5}$$

where $\delta(x-x')$ is the Dirac delta function, we obtain the Poisson integral formula for the Stokes problem in Ω:

$$\boldsymbol{u}=-\int_\Gamma\begin{bmatrix}\boldsymbol{g}(\boldsymbol{G}_1, Q_1)^T\\ \boldsymbol{g}(\boldsymbol{G}_2, Q_2)^T\end{bmatrix}\boldsymbol{u}_0\,ds,\tag{6}$$

where

$$\boldsymbol{g}(\boldsymbol{G}_i, Q_i)=\begin{bmatrix}\sigma_{11}(\boldsymbol{G}_i, Q_i) & \sigma_{12}(\boldsymbol{G}_i, Q_i)\\ \sigma_{21}(\boldsymbol{G}_i, Q_i) & \sigma_{22}(\boldsymbol{G}_i, Q_i)\end{bmatrix}\begin{bmatrix}n_1\\ n_2\end{bmatrix}_\Gamma,\quad i=1,\ 2.$$

From (6) and the corresponding formula for p, we can get the relation between $\boldsymbol{g}=(g_1, g_2)^T$ and $\boldsymbol{u}_0$:

$$\boldsymbol{g}=\mathscr{K}\boldsymbol{u}_0,\tag{7}$$