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INFINITE ELEMENT APPROXIMATION TO AXIAL SYMMETRIC STOKES FLOW*

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We considered in [1] the finite element approximation to axial symmetric Stokes flow in a bounded domain. The problem for the flow passing an obstacle in an unbounded domain is also frequently encountered. In this paper, we are going to give approximate solutions for this problem by an approach stated in [2]. An iterative method[3-5] is used to calculate the combined stiffness matrix.

§ 1. The Reduction to a System of Finite Algebraic Equations

Let us consider a rigid body in a 3-dimensional space, around which there is incompressible viscous fluid with steady velocity u. We assume that the flow at infinity is homogeneous with a velocity u_∞, and the Reynolds number is so small that the assumption of Stokes flow is acceptable. We can always replace u with $u-u_\infty$; therefore it is no harm to deem $u_\infty=0$. Now we give the classical formulation of the axial symmetric Stokes flow. Let $x=(x_1, x_2)\in R^2$, $R_+^2=\{x\in R^2;\ x_1>0\}$, and introduce in R^2 the polar coordinates (r, θ). Suppose there is a broken line Γ with end points at the x_2-axis and Ω is the exterior of Γ in R_+^2 (Fig. 1). Consider the following problem: to find $u(x)=(u_1(x), u_2(x))$, $p(x)$, satisfying

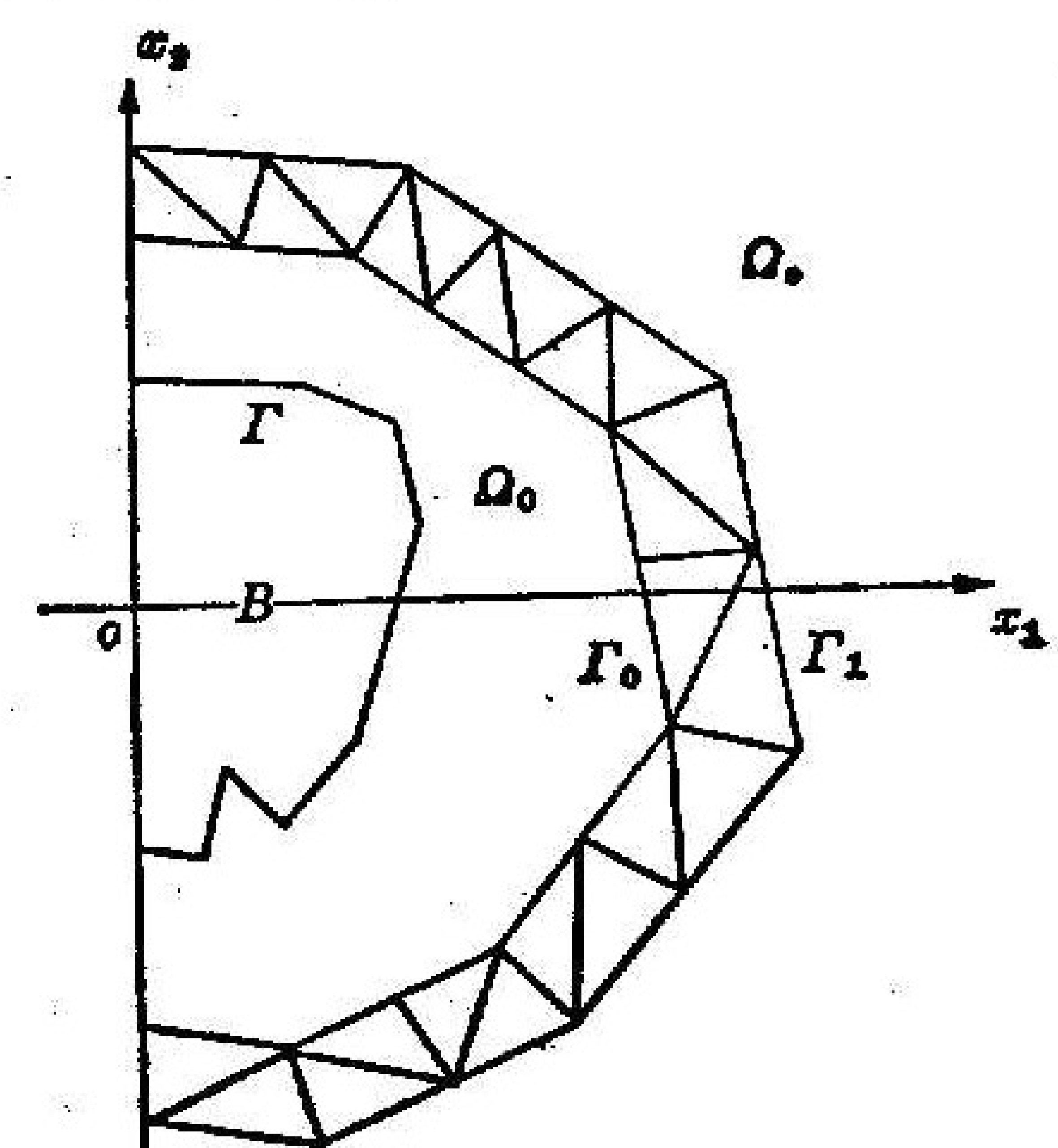

Fig. 1

$$\nu(-\nabla(x_1\nabla u_1)/x_1+u_1/x_1^2)+\partial p/\partial x_1=0, \quad x\in\Omega,$$

$$-\nu\nabla(x_1\nabla u_2)/x_1+\partial p/\partial x_2=0, \quad x\in\Omega,$$

$$\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)=0, \quad x\in\Omega,$$

$$u=u_*(x), \quad x\in\Gamma,$$

$$u_1=0, \quad x\in\partial\Omega\cap\{x_1=0\},$$

$$u=0,\ p=0, \quad |x|=\infty;$$

where ν is a positive constant and $u_*(x)$ is a known function.

We define some weighted Sobolev spaces for the above problem. There is no harm in assuming $|x|\geqslant\delta>0$ for every point x in Ω. The following semi-norm and

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norm

$$|f|_{m,\beta,\Omega}=\Big(\sum_{|\alpha|=m}\int_\Omega x_1|x|^{2(m-\beta)}|D^\alpha f|^2\,dx\Big)^{1/2},$$

$$\|f\|_{m,\beta,\Omega}=\Big(\sum_{i=0}^{m}|f|^2_{i,\beta,\Omega}\Big)^{1/2}$$

are defined and the corresponding Hilbert spaces are denoted by $Z^{m,\beta}(\Omega)$. We also define the norms as

$$|f|_{1,\beta,*,\Omega}=(|f|^2_{1,\beta,\Omega}+\|f/x_1\|^2_{0,\beta-1,\Omega})^{1/2},$$

$$\|f\|_{1,\beta,*,\Omega}=(|f|^2_{1,\beta,*,\Omega}+\|f\|^2_{0,\beta,\Omega})^{1/2}.$$

The corresponding Hilbert spaces are denoted by $Z^{1,\beta}_*(\Omega)$, and $Z^{2,\beta}_+(\Omega)$ is a set such that $f\in Z^{2,\beta}_+(\Omega)$ if and only if $f\in Z^{1,\beta}_*(\Omega)$ and $\|D^{(0,2)}f/x_1\|_{0,\beta-2,\Omega}$ is finite. The above definitions are equivalent to that in [1] when Ω is bounded.

Let $H(\Omega)=Z^{1,1}_*(\Omega)\times Z^{1,1}(\Omega)$, $H_0(\Omega)=\{u\in H(\Omega);\ u|_{\partial\Omega\setminus(x_1=0)}=0\}$. Consider the bilinear form

$$a(u,v)_\Omega=\nu\int_\Omega x_1(\nabla u_1\cdot\nabla v_1+\nabla u_2\cdot\nabla v_2+u_1v_1/x_1^2)dx,\quad u,v\in H(\Omega),\tag{1}$$

defined in $H(\Omega)\times H(\Omega)$, and the bilinear form

$$b(v,p)_\Omega=-\int_\Omega p\Big\{\frac{\partial}{\partial x_1}(x_1v_1)+\frac{\partial}{\partial x_2}(x_1v_2)\Big\}dx,\quad v\in H(\Omega),\ p\in Z^{0,0}(\Omega),\tag{2}$$

defined in $H(\Omega)\times Z^{0,0}(\Omega)$. The definitions for bilinear forms with respect to other domains are similar. Let $H(\Gamma)$ be the trace space of $H(\Omega)$ on Γ; then the weak formulation for the original problem is: to find $(u,p)\in H(\Omega)\times Z^{0,0}(\Omega)$, such that

$$a(u,v)_\Omega+b(v,p)_\Omega=0,\quad \forall v\in H_0(\Omega),\tag{3}$$

$$b(u,q)_\Omega=0,\quad \forall q\in Z^{0,0}(\Omega),\tag{4}$$

$$u|_\Gamma=u_*,\tag{5}$$

where $u_*\in H(\Gamma)$. The solution of this problem exists and is unique.

Let us consider the infinite element approximation to problem (3)—(5). We construct a broken line Γ_0: $r=r_0(\theta)$, $|\theta|<\frac{\pi}{2}$, which divides Ω into Ω_* and Ω_0, where Ω_0 lies between Γ and Γ_0 and Ω_* is the exterior of Γ_0. We assume that Γ_0 is star-shaped with respect to the point 0, i.e. each ray from the point 0 intersects Γ_0 at most at one point. Especially, it may happen that $\Gamma_0=\Gamma$; then Ω_0 is empty.

Taking a constant $\xi>1$, we construct similar curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots$ of Γ_0 with 0 as the center and $\xi, \xi^2, \cdots, \xi^k, \cdots$ as constants of proportionality. Let

$$\Omega_k=\Big\{(r,\theta);\ \xi^{k-1}r_0(\theta)<r<\xi^k r_0(\theta),\ |\theta|<\frac{\pi}{2}\Big\},$$

$$\Omega_{*,k}=\Big\{(r,\theta);\ r_0(\theta)<r<\xi^k r_0(\theta),\ |\theta|<\frac{\pi}{2}\Big\}.$$

Domain Ω is triangulated in such a way that $\Omega_0, \Omega_1, \Omega_2, \cdots$ consist exactly of finite triangular elements, and the triangulation of $\Omega_1, \Omega_2, \cdots, \Omega_k, \cdots$ is geometrically similar. In each element, second order interpolation is used for u and p is constant, just as in [1]. For definiteness, we assume that each subdomain Ω_k is divided into