THE CONVERGENCE OF NUMERICAL METHOD FOR NONLINEAR SCHRÖDINGER EQUATION*

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§1. Introduction

In the past few years interest has substantially increased in the solutions of partial differential equations governing nonlinear waves in dispersive media, while considerable literature has grown dealing with the numerical approximations of such problems. One of nonlinear wave equations is nonlinear Schrödinger equation whose solution is a complex field governing the evolution of any weakly nonlinear, strongly dispersive, almost monochromatic wave (see Zakharov (1968), Hasimoto and Ono (1972), Davey (1972) and Yuen and Lake (1980)). The pure initial value problem was exactly solved by Zakharov and Shabat (1972) using the inverse scattering method when the initial condition vanishes for sufficiently large $|x|$. For more general initial conditions the theoretical solution of the nonlinear Schrödinger equation is unknown. From the numerical point of view, Ablowitz and Ladik (1976) employed a difference scheme for the numerical solution of the nonlinear Schrödinger equation. Other methods were given by Yuen and Lake (1975), Kuo Pen-yu (1976), Yuen and Ferguson (1978), Yuen and Lake (1980) and Defour, Forten and Payne (1981). Recently Griffiths, Mitchell and Morris (1982) proposed a prediction-correction scheme which does not need nonlinear iteration. If we choose the parameter suitably in that scheme, then the scheme is stable and has high order accuracy. The numerical results showed the advantage of that method. For the strict error estimations, Kuo Pen-yu (1979) gave a proof for the semi-discrete scheme. Recently Zhu You-lan (1983) considered an implicit scheme and gave its convergence.

This paper is devoted to the convergence of some numerical methods such as the Crank-Nicolson method and the prediction-correction method, the finite difference scheme and the Galerkin method.

§2. Crank-Nicolson Method

We consider the following problem

$$\begin{cases} i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\alpha w|w|^2=0, & x\in \mathbf{R},\ 0<t<T, \\ w(x,0)=w_0(x), & x\in \mathbf{R}, \end{cases} \tag{1}$$

where $\alpha>0$. We suppose that for all $t\geqslant 0$, $w(x,t)\in L^2(\mathbf{R})$ and for all $x\in\mathbf{R}$, $0\leqslant t\leqslant T$,

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$w(x, t)$ is bounded,

$$\lim_{|x|\to\infty} w(x, t)=0, \quad \lim_{|x|\to\infty} \frac{\partial w}{\partial x}(x, t)=0.$$

Let $w(x, t)=u(x, t)+iv(x, t)$ where $u(x, t)$ and $v(x, t)$ are real value functions, then (1) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial^2 v}{\partial x^2}+\alpha v|w|^2=0, & x\in\mathrm{R},\ 0<t\leqslant T, \\ \dfrac{\partial v}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}-\alpha u|w|^2=0, & x\in\mathrm{R},\ 0<t\leqslant T, \\ u(x, 0)=u_0(x), & x\in\mathrm{R}, \\ v(x, 0)=v_0(x), & x\in\mathrm{R}. \end{cases} \tag{2}$$

Let

$$\|w(t)\|^2_{L^2}=\int_{\mathrm{R}}[|u(x, t)|^2+|v(x, t)|^2]dx,$$

then we have

$$\|w(t)\|^2_{L^2}=\|w_0\|^2_{L^2}, \tag{3}$$

and

$$\left\|\frac{\partial w(t)}{\partial x}\right\|^2_{L^2}-\frac{\alpha}{2}\|w(t)\|^4_{L^4}=\left\|\frac{\partial w_0}{\partial x}\right\|^2_{L^2}-\frac{\alpha}{2}\|w_0\|^4_{L^4}. \tag{4}$$

For the numerical solution of (2), we take h and k to be the mesh spacings for variables x and t respectively, $x_j=jh$, $j=0, \pm1, \pm2, \cdots$, $\mathrm{R}_h=\{x_j\}$. Let $\eta^n(x)$ be the value of the function η at $x\in\mathrm{R}_h$ and $t=nk$. We introduce the following notations

$$\eta^n_t(x)=\frac{1}{k}(\eta^{n+1}(x)-\eta^n(x)),$$

$$\eta^n_x(x)=\frac{1}{h}(\eta^n(x+h)-\eta^n(x)),$$

$$\eta^n_{\bar{x}}(x)=\eta^n_x(x-h),$$

and

$$\eta^n_{x\bar{x}}(x)=[\eta^n_x(x)]_{\bar{x}}.$$

We define the discrete scalar product and the norm as follows

$$(\eta^n, \xi^n)=h\sum_{x\in\mathrm{R}_h}\eta^n(x)\xi^n(x),$$

$$\|\eta^n\|^2=(\eta^n, \eta^n).$$

It can be proved that

$$(\eta^n_{x\bar{x}}, \xi^n)=(\xi^n_{x\bar{x}}, \eta^n), \tag{5}$$

$$(\eta^n_t, \eta^n+\eta^{n+1})=[\|\eta^n\|^2]_t, \tag{6}$$

$$2(\eta^n_t, \eta^n)=[\|\eta^n\|^2]_t-\tau\|\eta^n_t\|^2, \tag{7}$$

and

$$\|\eta^n\xi^n\|\leqslant\frac{1}{h}\|\eta^n\|\|\xi^n\|. \tag{8}$$

Let $U^n(x)$, $V^n(x)$ and $W^n(x)$ be the approximations of $u^n(x)$, $v^n(x)$ and $w^n(x)$ respectively. The Crank-Nicolson scheme for solving (2) is