ON THE CONVERGENCE OF QUASI-CONFORMING ELEMENTS FOR LINEAR ELASTICITY PROBLEMS*

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Abstract

Continuing the work in [1, 2], we discuss the convergence conditions and the error estimates of quasi-conforming elements for linear elasticity problems. Some results about curved elements for second-order boundary value problems are also given.

§ 1. Introduction

The quasi-conforming element method is very efficient and successful for elliptic problems (see [1—4]). The mathematical foundation of this method has been established in [1, 2], and several plate bending elements have been shown to be convergent. In the present paper, we mainly discuss the convergence conditions of quasi-conforming elements for linear elasticity problems and their error estimates. In addition we give some results about curved elements for second-order boundary value problems.

Let Ω be a bounded connected domain in R^n with Lipschitz-continuous boundary $\partial\Omega$. For each $v=(v_1, v_2, \cdots, v_n)$ in $(H^1(\Omega))^n$, we set

$$\begin{cases} \varepsilon_{ij}(v)=\varepsilon_{ji}(v)=(\partial_i v_j+\partial_j v_i)/2, \quad 1\leqslant i,\ j\leqslant n, \\ \sigma_{ij}(v)=\sigma_{ji}(v)=\lambda\Big(\sum_{l=1}^{n} \varepsilon_{ll}(v)\Big)\delta_{ij}+2\mu\varepsilon_{ij}(v), \quad 1\leqslant i,\ j\leqslant n, \end{cases} \tag{1.1}$$

where δ_{ij} is Kronecker's symbol, and λ and μ are two positive constants.

Consider the boundary value problem of linear elasticity:

$$\begin{cases} -\sum_{j=1}^{n} \partial_j\sigma_{ij}(u)=f_i, \quad i=1, \cdots, n, \text{ in } \Omega, \\ u|_{\partial\Omega}=0. \end{cases} \tag{1.2}$$

In order to discuss this problem, we need some notations. Let $H=\{\varLambda=(V_i, E_{ij})\,|\,V_i, E_{ij}\in L^2(\Omega), 1\leqslant i,\ j\leqslant n \text{ with } E_{ij}=E_{ji}\}$. For $v=(v_1, \cdots, v_n)$ in $(H^1(\Omega))^n$, define $Tv=(v_j,\ \varepsilon_{ij}(v))$. Then $T(H^1(\Omega))^n$ is a subspace of H. And for each $\varLambda$ in H, define

$$\sigma_{ij}(\varLambda)=\sigma_{ji}(\varLambda)=\lambda\Big(\sum_{l=1}^{n} E_{ll}\Big)\delta_{ij}+2\mu E_{ij}, \quad 1\leqslant i,\ j\leqslant n. \tag{1.3}$$

We define the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ on space H as

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follows:

$$a(\Lambda, \overline{\Lambda}) = \sum_{i,j=1}^{n} \int_{\Omega} \sigma_{ij}(\Lambda) \overline{E}_{ij} dx, \quad \forall \Lambda, \overline{\Lambda} \in H, \tag{1.4}$$

$$f(\Lambda) = \int_{\Omega} \sum_{i=1}^{n} f_i V_i dx, \quad \forall \Lambda \in H. \tag{1.5}$$

Then problem (1.2) is equivalent to the following variational problem:

$$u \in (H_0^1(\Omega))^n, \ a(Tu, Tv) = f(Tv), \quad \forall v \in (H_0^1(\Omega))^n. \tag{1.6}$$

Let $\{U_h\}$ be a family of finite dimensional subspaces in H for parameter h with $h>0$ and $h\to 0$. According to [1, 2], the finite element approximation with multiple sets of functions to problem (1.6) is the following problem

$$\Lambda_h^* \in U_h, \ a(\Lambda_h^*, \Lambda_h) = f(\Lambda_h), \quad \Lambda_h \in U_h. \tag{1.7}$$

The work in this paper is mainly devoted to the following subjects: i) the method for constructing spaces U_h using quasi-conforming elements, ii) the existence, uniqueness and convergence of the solution of problem (1.7).

§ 2. Quasi-Conforming Element Method

In this section we discuss how to construct spaces U_h by the quasi-conforming element method. From now on, we assume that Ω is a polyhedroid domain in R^n, and let K_h be a finite subdivision of Ω for each h with the properties K1 and K2:

K1. For every element K in K_h, K is an n-simplex (or n-parallelotope) and

$$\bigcup_{K \in K_h} = \overline{\Omega}.$$

K2. For every two different elements K and K' in K_h, $K \cap K'$ is an empty set or a common face of K and K'[1].

Let t be a positive number. For any n-simplex (or n-parallelotope) K, we give two linear interpolation operators Π_K: $H^t(K) \to L^2(K)$ and $\Pi_{\partial K}$: $H^t(K) \to L^2(\partial K)$ and some finite dimensional spaces consisting of polynomials, say N_K^{ij}, $1 \leqslant i, j \leqslant n$, with $N_K^{ij} = N_K^{ji}$. For each $v = (v_1, \cdots, v_n)$ in $(H^t(K))^n$, we define $E_K^{ij}(v)$ in N_K^{ij}, $1 \leqslant i, j \leqslant n$, by the following equations:

$$2\int_K p E_K^{ij}(v) dx = \int_{\partial K} p(\Pi_{\partial K} v_i N_j + \Pi_{\partial K} v_j N_i) ds$$
$$- \int_K (\partial_j p \Pi_K v_i + \partial_i p \Pi_K v_j) dx, \quad 1 \leqslant i, j \leqslant n, \ \forall p \in N_K^{ij}, \tag{2.1}$$

where $N = (N_1, \cdots, N_n)^T$ is the unit outward normal of ∂K. Then we use $\Pi_K v_j$ and $E_K^{ij}(v)$ to approximate v_j and $s_{ij}(v)$ respectively.

Now we are in a position to construct spaces U_h. Define an operator Π_h: $(H^t(\Omega))^n \to H$ such that, for each v in $(H^t(\Omega))^n$, $\Pi_h v = (\Pi_h^j v, E_h^{ij} v)$ satisfies

$$\begin{cases} \Pi_h^j v|_K = \Pi_K(v_j|_K), & 1 \leqslant j \leqslant n, \ K \in K_h, \\ E_h^{ij} v|_K = E_K^{ij}(v|_K), & 1 \leqslant i, j \leqslant n, \ K \in K_h. \end{cases} \tag{2.2}$$

Then the spaces U_h are obtained by setting $U_h = \Pi_h(H^t(\Omega) \cap H_0^1(\Omega))^n$; this is the

1) A subset F is said to be a face of K if there exists a supporting hyper-plane $\mathscr{P}$ of K such that $F = K \cap \mathscr{P}$.