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AN ANALYSIS OF PENALTY-NONCONFORMING FINITE ELEMENT METHOD FOR STOKES EQUATIONS*

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Abstract

In this paper, the penalty-nonconforming finite element method for Stokes equations is considered. An abstract error estimate is given. For Crouzeix-Raviart nonconforming triangular elements, in particular, the analysis shows that the "reduced integration" technique is not necessary in the integration of the penalty term on each element. It means that a loss of precision is avoided in this penalty method.

§1. Introduction

We consider the numerical analysis of a class of finite element method for Stokesian flow problems of the type

$$\begin{cases} -\mu\Delta \boldsymbol{u}+\nabla p=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is the viscosity, $\boldsymbol{u}=(u_1, \cdots, u_n)$ is the velocity field, p is the pressure, $\boldsymbol{f}$ is the body force density, and Ω is an open bounded domain in $\mathbb{R}^n$. $\partial\Omega$ is the boundary of Ω satisfying the Lipschitz condition.

As usual, let $H^m(\Omega)$, $H_0^m(\Omega)$ denote the Sobolev spaces with norm $\|\cdot\|_{m,\Omega}$, and $V=(H_0^1(\Omega))^n$, $M=\{q\in L^2(\Omega), \int_\Omega q\,dx=0\}$. Then the boundary value problem (1.1) is equivalent to the following variational problem:

Find $(\boldsymbol{u}, p)\in V\times M$, such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, & \forall \boldsymbol{v}\in V, \\ b(\boldsymbol{u}, q)=0, & \forall q\in M, \end{cases} \tag{1.2}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v})=\mu\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,dx,$$

$$b(\boldsymbol{v}, q)=-\int_\Omega q(\operatorname{div} \boldsymbol{v})dx=-(\operatorname{div} \boldsymbol{v}, q),$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

A direct finite-element approximation of problem (1.2) leads to the so-called

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mixed finite element methods using conforming and nonconforming finite elements which had been studied extensively, see [1]—[5]. An alternative formulation of (1.2) is provided by the exterior penalties where (1.2) is replaced by a family of perturbations consisting of unconstrained problems depending on a penalty parameter $\varepsilon>0$.

Let ε be an arbitrary positive number. Then a penalty approximation of the variational problem (1.2) consists of finding $(\boldsymbol{u}_\varepsilon, p_\varepsilon)\in V\times M$, such that

$$\begin{cases} a(\boldsymbol{u}_\varepsilon, \boldsymbol{v})+b(\boldsymbol{v}, p_\varepsilon)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, & \forall \boldsymbol{v}\in V, \\ b(\boldsymbol{u}_\varepsilon, q)-\varepsilon(p_\varepsilon, q)=0, & \forall q\in M. \end{cases} \tag{1.3}$$

For any $\boldsymbol{v}\in V$, we have

$$\int_\Omega \operatorname{div} \boldsymbol{v}\, dx=0;$$

then we can eliminate the pressure p_ε from the last equation and get

$$p_\varepsilon=-\frac{1}{\varepsilon}\operatorname{div}\boldsymbol{u}_\varepsilon, \quad \text{in } \Omega. \tag{1.4}$$

Finally, we obtain the penalty approximation of the variational problem (1.2) containing only unknown functions $\boldsymbol{u}_\varepsilon$:

$$a(\boldsymbol{u}_\varepsilon, \boldsymbol{v})+\varepsilon^{-1}(\operatorname{div}\boldsymbol{u}_\varepsilon, \operatorname{div}\boldsymbol{v})=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in V. \tag{1.5}$$

The variational problem (1.5) and (1.4) is equivalent to problem (1.3). The significant advantage in the penalty variational problem (1.5) is that the pressure does not appear explicitly in the variational formulation; hence the corresponding finite element schemes can be constructed to have fewer unknowns than the standard mixed methods.

Finite element methods based on (1.5) have been proposed by several authors[5—9], who on the basis of numerical experiments, have determined that it is necessary to use reduced integration of the penalty terms in formulation (1.5) in order to obtain physically reasonable results. These reduced-integration-penalty schemes also have been studied mathematically by several authors. In particular, we refer to the work of Oden, Kikuchi and Song[10].

In this paper, nonconforming finite elements are applied to penalty finite element methods for Stokes equations. Moreover, an abstract error estimate is given. For nonconforming triangular elements, in particular, the reduced integration technique is not necessary. It means that the integration of the penalty term on each element is required to integrate exactly.

§ 2. Nonconforming Finite Element Approximation

First of all, we recall the basic convergence theorem for penalty problem (1.5).

Theorem 2.1. *Given $\varepsilon>0$, let $\boldsymbol{u}_\varepsilon\in V$ be the solution of (1.5) and let p_ε be the function given by (1.4). Then $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges strongly to solution $(\boldsymbol{u}, p)$ of (1.2) in $V\times M$ as $\varepsilon\to 0$. Moreover, the following estimates hold*

$$\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_V+|p-p_\varepsilon|_M\le C\varepsilon,$$