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HIGH ORDER APPROXIMATION OF ONE-WAY WAVE EQUATIONS*

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Abstract

In this article the high order approximations of the one-way wave equations are discussed. The approximate dispersion relations are expressed in explicit form of sums of simple fractions. By introducing new functions, the high order approximations of the one-way wave equations are put into the form of systems of lower order equations. The initial-boundary value problem of these systems which corresponds to the migration problem in seismic prospecting is discussed. The energy estimates for their solutions are obtained.

Introduction

The wave equation describes waves propagating in all directions. The equation, which describes only the down-going (or up-coming) waves propagating in the positive (or negative) direction of z, is called the one-way wave equation. In the one-dimensional case the one-way wave equations are the simple wave equations

$$\left(\frac{\partial}{\partial z}\pm\frac{1}{c}\frac{\partial}{\partial t}\right)p=0, \tag{1}$$

the general solutions of which are $f(t\mp z/c)$. The constant c is the velocity of propagation. In the two-dimensional case the one-way wave equations

$$L_{\pm}p=0 \tag{2}$$

describe the waves $f(t-(\alpha z+\beta x)/c)$ for all $\alpha\geqslant 0$ (or $\alpha\leqslant 0$), where α, β satisfy $\alpha^2+\beta^2=1$. The operators $L_{\pm}$ can be defined in terms of the Fourier transforms as pseudo-differential operators. For practical application it is necessary to derive their approximations that have local character. Such approximations are obtained in [1, 2, 3, 4] as the artificial boundary conditions for the wave equation, and also in [5, 6, 7] as the basic equations for migration in seismic prospecting.

The n-th order approximation obtained in the papers mentioned above is the $(n+1)$-th order P. D. E.. It is difficult to apply them for computation when $n\geqslant 2$. One of our purposes is to derive a new form of these approximations which is more convenient for numerical application. First, we derive the explicit expressions of the approximate dispersion relations, based on which the approximations of the one-way wave equations can be obtained. Then we derive systems of lower order P. D. E. as new forms of these approximations. Finally, we discuss the initial-boundary value problem (migration problem in seismic prospecting) of these systems and obtain the energy estimates for their solutions.

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1. Approximate Dispersion Relations

Consider the wave equation

$$\frac{\partial^2 p}{\partial z^2}+\frac{\partial^2 p}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=0, \tag{1.1}$$

where the constant c is the velocity of propagation. Suppose that the Fourier transform $\hat{p}$ of the solution p exists

$$\hat{p}(z;\ K_x,\ \omega)=\frac{1}{2\pi}\iint \exp(i\omega t+iK_x x)p(z,\ x,\ t)dx\,dt;$$

then $\hat{p}$ satisfies the wave equation in frequency domain

$$\left(\frac{d^2}{dz^2}+K_z^2\right)\hat{p}=\left(\frac{d}{dz}-iK_+\right)\left(\frac{d}{dz}-iK_-\right)\hat{p}=0, \tag{1.2}$$

where

$$K_\pm=\pm K_z=\pm K\sqrt{1-K_x^2/K^2}, \tag{1.3}$$

in which $K=\omega/C$. (1.3) is called the dispersion relation of wave equation (1.1). The one-way wave equations in frequency domain are the following

$$\begin{aligned}&\left(\frac{d}{dz}-iK_+\right)\hat{p}=0 \quad \text{for down-going wave,}\\ &\left(\frac{d}{dz}-iK_-\right)\hat{p}=0 \quad \text{for up-coming wave.}\end{aligned} \tag{1.4}$$

From (1.4), we can see that the inverse Fourier transform p of $\hat{p}$ satisfies the equation

$$L_\pm p=\left(\frac{\partial}{\partial z}-\mathscr{K}_\pm\right)p=0 \tag{1.5}$$

with the pseudo-differential operators $\mathscr{K}_\pm$, the symbols of which are $K_\pm$.

The objective of this section is to derive the rational fraction approximations of $K_\pm$.

Let

$$S=(\mp K_\pm+K)/K_x, \quad r=K/K_x. \tag{1.6}$$

Then from (1.3) we see that S satisfies

$$S^2-2rS+1=0. \tag{1.7}$$

The smaller root S_∞ of (1.7) can be approximated by S_n, which are defined by the recursion relation[2]

$$S_0=0, \quad S_{n+1}=1/(2r-S_n). \tag{1.8}$$

Lemma 1. (1) *For any $r>1$, the sequence S_n is monotonically increasing, and*

$$\lim_{n\to\infty} S_n=S_\infty=r-\sqrt{r^2-1}<1, \tag{1.9}$$

$$S_\infty-S_n=0(1/r^{2n+1}). \tag{1.10}$$

(2)

$$S_n(r)=Q_{n-1}(r)/Q_n(r), \tag{1.11}$$

where $Q_n(r)$ is the n-th order Tchebyschev polynomial of the second kind, i.e.

$$Q_n(r)=2^n\prod_{l=1}^{n}(r-\alpha_{n,l}), \quad \alpha_{n,l}=\cos(l\pi/n+1). \tag{1.12}$$