NUMERICAL SOLUTION OF THE REACTION-DIFFUSION EQUATION*1)

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In this paper, we consider the numerical solution for the equation

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0.$$

A finite difference scheme and the basic error equality are given. Then the error estimations are proved for the periodic problem with $\nu(x, t)\geqslant 0$, the first and second boundary value problems with $\nu(x, t)\geqslant \nu_0>0$, and for $\nu(U)\geqslant\nu_0>0$. Under some conditions such estimations imply the stabilities and convergences of the schemes.

§ 1. Introduction

In one-dimensional space, the reaction-diffusion equation is the following

$$\frac{\partial U}{\partial t}+M(x, t, U)\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0.$$

Much work has been done to solve this equation (see [1]). On the other hand some authors worked at error estimations. But there are still some unsolved problems:

(i) In [2], the stability is taken as the boundedness of the solution. But in fact the boundedness of the solution of a non-linear scheme is not uniform with the stability. Besides it is supposed that

$$\nu(x, t, U)\geqslant\frac{K}{2}|M(x, t, U)|, \quad K>0.$$

So the following important case is excluded:

$$M(x, t, U)=U, \quad \nu=\text{positive constant}.$$

(ii) In [3], the author considered the following case:

$$M(x, t, U)=U^p, \quad p\geqslant 1,$$

but only for the periodic problem with $\nu(x, t, U)=$ positive constant.

(iii) Recently the author[4] studied the numerical solution of Burger's equation and used the same technique for the reaction-diffusion equation, but only for some special cases (see [5]).

This paper is concerned with a general problem, i.e.

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0, \quad 0\leqslant x\leqslant 1,\ t>0, \quad (1.1)$$

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1) This work is done on the basis of the proposition of Professor A. R. Mitchell when the author visited Dundee University in May, 1980.

where $\nu(x, t, U) \geqslant 0$.

The technique used is to estimate the index of generalized stability (see [6—8]). In section 2, we give some notations and lemmas. In section 3, a scheme is constructed and the basic error equality is proved. In section 4, we give a strict error estimation for the periodic problem with $\nu(x, t) \geqslant 0$. In sections 5 and 6, we prove strict error estimations with the first or second boundary value conditions. In section 7, we consider the case $\nu(x, t, U) = \nu(U) > 0$.

§ 2. Notations and Lemmas

Let h and τ be the mesh spacing of variables x and t respectively. The mesh point is $(jh, k\tau)$.

$$u_x(jh, k\tau) = \frac{1}{h}[u(jh+h, k\tau) - u(jh, k\tau)],$$

$$u_{\bar{x}}(jh, k\tau) = \frac{1}{h}[u(jh, k\tau) - u(jh-h, k\tau)],$$

$$u_{\hat{x}}(jh, k\tau) = \frac{1}{2h}[u(jh+h, k\tau) - u(jh-h, k\tau)],$$

$$\Delta_h^{\nu(jh, k\tau, u(jh, k\tau))} u(jh, k\tau) = \frac{1}{2}[\nu(jh, k\tau, u(jh, k\tau)) u_{\bar{x}}(jh, k\tau)]_x + \frac{1}{2}[\nu(jh, k\tau, u(jh, k\tau)) u_x(jh, k\tau)]_{\bar{x}},$$

$$u_t(jh, k\tau) = \frac{1}{\tau}[u(jh, k\tau+\tau) - u(jh, k\tau)].$$

We define

$$(u(k\tau), v(k\tau)) = h \sum_{j=1}^{N-1} u(jh, k\tau) v(jh, k\tau),$$

$$\|u(k\tau)\|^2 = (u(k\tau), u(k\tau)),$$

$$|u(k\tau)|^2_{1, \nu(k\tau, u(k\tau))} = \frac{h}{2} \sum_{j=1}^{N-1} \nu(jh, k\tau, u(jh, k\tau)) [u_x^2(jh, k\tau) + u_{\bar{x}}^2(jh, k\tau)].$$

If $\nu(jh, k\tau, u(jh, k\tau)) \equiv 1$, then $|u(k\tau)|^2_{1, \nu(k\tau, u(k\tau))}$ is denoted by $|u(k\tau)|^2_1$ for simplicity.

Lemma 1.

$$2(u(k\tau), u_t(k\tau)) = \|u(k\tau)\|^2_t - \tau \|u_t(k\tau)\|^2.$$

Lemma 2.

$$(u(k\tau), \Delta_h^{\nu(k\tau, u(k\tau))} u(k\tau)) + |u(k\tau)|^2_{1, \nu(k\tau, u(k\tau))} = D_1,$$

where

$$\begin{aligned} D_1 = & \frac{1}{2} u_{\bar{x}}(Nh, k\tau)[\nu(Nh, k\tau, u(Nh, k\tau)) u(Nh-h, k\tau) \\ & + \nu(Nh-h, k\tau, u(Nh-h, k\tau)) u(Nh, k\tau)] \\ & - \frac{1}{2} u_x(0, k\tau)[\nu(h, k\tau, u(h, k\tau)) u(0, k\tau) \\ & + \nu(0, k\tau, u(0, k\tau)) u(h, k\tau)]. \end{aligned}$$

Proof. From Abel's formula we have

$$(\eta_x(k\tau), \xi(k\tau)) + (\xi_{\bar{x}}(k\tau), \eta(k\tau))$$