A NEW UNIFORMLY CONVERGENT ITERATIVE METHOD BY INTERPOLATION, WHERE ERROR DECREASES MONOTONICALLY*

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Introduction

For an arbitrarily refined system of knots, polynomial interpolation does not guarantee the convergency. Hence grew the piecewise interpolation. But the currently widely spread spline interpolation has some shortcomings in practical computation. Adding any new knot, we will have to solve a new linear system of equations. Besides, spline functions always possess some degree of smoothness, and the smooth spline interpolation is not a suitable means for approximation of a less smooth function.

In this paper, we introduce a so called "regenerating kernel" $R_x(y)$, with which to derive a formula of interpolation, and construct a new simple iterative method. Having got some approximation, we put a new knot in each step, interpolate the error function and add the result to the previous approximation. In this way we get another approximation. The formula is very simple and feasible for computer use.

We have proven:

1) With a new knot, the error of approximation decreases in the sence of Sobolev norm

$$\|u\|=\int_a^b u^2\,dx+\int_a^b u'^2\,dx\Big)^{\frac{1}{2}}.$$

2) For an arbitrarily thickened knot system, the iterative process converges uniformly.

Actual computation has verified the theory. Error decreased monotonically. When the knot system was refined, accuracy increased considerably.

For the Lamp function, which has a turning point (derivative discontinuous at the origin), our result is better than that obtained by cubic spline.

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§ 1

Definition.

$$R_x(y)=\frac{1}{2\,\mathrm{sh}(b-a)}[\mathrm{ch}(x+y-a-b)+\mathrm{ch}(|x-y|-b+a)],\quad a\leqslant x\leqslant b,\ a\leqslant y\leqslant b.$$

The following are evident by definition.

1. $R_x(y)=R_y(x)>0.$ (1)

2. $R_x(y)$ satisfies the differential equation

$$-\frac{d^2R_x(y)}{dy^2}+R_x(y)=0,\quad x\neq y. \tag{2}$$

Hence

$$\frac{d^2R_x(y)}{dy^2}>0,\quad x\neq y. \tag{3}$$

3. $$\begin{cases}\dfrac{dR_x(y)}{dy}>0, & a<y<x,\\ \dfrac{dR_x(y)}{dy}<0, & x<y<b.\end{cases} \tag{4}$$

4. $$\left.\frac{dR_x(y)}{dy}\right|_{y=a+0}=\left.\frac{dR_x(y)}{dy}\right|_{y=b-0}=0,\quad a<x<b. \tag{5}$$

5. $$\left.\frac{dR_x(y)}{dy}\right|_{y=x-0}-\left.\frac{dR_x(y)}{dy}\right|_{y=x+0}=1,\quad a<x<b; \tag{6}$$

$$-\left.\frac{dR_a(y)}{dy}\right|_{y=a+0}=\left.\frac{dR_b(y)}{dy}\right|_{y=b-0}=1. \tag{6'}$$

6. Using (3), (4), (6), it is easy to prove

$$\left|\frac{dR_x(y)}{dy}\right|\leqslant 1. \tag{7}$$

7. $$\max R_x(y)=\max R_a(y)=R_a(a)=R_b(b)=\frac{\mathrm{ch}(b-a)}{\mathrm{sh}(b-a)}, \tag{8}$$

$$\min R_x(y)=R_a(b)=R_b(a)=\frac{1}{\mathrm{sh}(b-a)}, \tag{9}$$

$$\min R_y(y)=\frac{1+\mathrm{ch}(b-a)}{2\,\mathrm{sh}(b-a)},\quad y=\frac{a+b}{2}. \tag{10}$$

8. Let $W_2^1=\{u\,|\,u$ absolutely continuous, $u'\in L^2[a,b]\}$. For $u, v\in W_2^1$, we define the inner product

$$(u,v)=\int_a^b uv\,dx+\int_a^b u'v'\,dx.$$

Now, we verify

$$(R_x(\cdot),u(\cdot))=u(x),\quad a\leqslant x\leqslant b. \tag{11}$$

For $a<x<b$,

$$\begin{aligned}(R_x(y),u(y))&=\left(\int_a^{x-\varepsilon}+\int_{x-\varepsilon}^{x+\varepsilon}+\int_{x+\varepsilon}^b\right)R_x(y)u(y)\,dy+\left(\int_a^{x-\varepsilon}+\int_{x-\varepsilon}^{x+\varepsilon}+\int_{x+\varepsilon}^b\right)R_x'(y)u'(y)\,dy\\&=\int_a^{x-\varepsilon}(-R_x''(y)+R_x(y))u(y)dy+\int_{x+\varepsilon}^b(-R_x''(y)+R_x(y))u(y)dy\end{aligned}$$