MATHEMATICAL ASPECT OF OPTIMAL CONTROL FINITE ELEMENT METHOD FOR NAVIER-STOKES PROBLEMS*

LI KAI-TAI (李开泰) HUANG AI-XIANG (黄艾香)

(Department of Mathematics, Xi'an Jiaotong University, Xi'an, China)

Abstract

This study deals with the theoretical basis of optimal control methods in primitive variable formulation and penalty function formulation of Navier-Stokes problems. Numerical examples demonstrating application are provided.

1. Introduction

The finite element formulation of the Navier-Stokes equation governing the flow of a viscous incompressible fluid can be classified into five basic categories: (1) primitive variable formulation (or velocity-pressure formulation), (2) penalty function formulation, (3) stream function formulation, (4) stream function-vorticity formulation, and (5) optimal control formulation. Each of them has relative advantages and disadvantages. These formulations differ mainly in the way the incompressibility condition is included in the formulation.

The optimal control formulation is to minimize the energy functional by the introduction of the state vector, which is a solution of a Stokes problem. The incompressibility condition is treated as a constraint or is eliminated to introduce the penalty function.

2. Optimal Control Formulation

We consider the following boundary value problem of the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}|_{\Gamma_1}=0,\ \left(\nu\dfrac{\partial\boldsymbol{u}}{\partial n}-p\boldsymbol{n}\right)_{\Gamma_2}=\boldsymbol{g} & \text{on } \partial\Omega=\Gamma=\Gamma_1\cup\Gamma_2, \end{cases} \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity of the fluid, p the pressure, Ω a bounded domain of R^n with a Lipschitz continuous boundary.

We introduce the Sobolev space $X=[H^1(\Omega)]^n$ with the norm

$$\|\boldsymbol{u}\|_1^2=\sum_{i=1}^n\|u_i\|_1^2, \quad \forall\boldsymbol{u}\in X,$$

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Let $X_0=[H_0^1(\Omega)]^n$, $V=\{\boldsymbol{u}|\boldsymbol{u}\in X,\ \boldsymbol{u}|_{\Gamma_1}=0\}$, $V_0=\{\boldsymbol{u}|\boldsymbol{u}\in V,\ \operatorname{div}\boldsymbol{u}=0\}$; so $X\subset V\subset X$ and $V_0\subset V$.

We also introduce linear, bilinear and trilinear functional:

$$\begin{aligned}
&a_0(\boldsymbol{u},\ \boldsymbol{v})=\nu(\nabla\boldsymbol{u},\ \nabla\boldsymbol{v})=\nu\sum_{i,j=1}^n\int_\Omega\frac{\partial u^i}{\partial x^j}\,\frac{\partial v^i}{\partial x^j}\,dx,\\
&G(\boldsymbol{u},\ \boldsymbol{v})=(\operatorname{div}\boldsymbol{u},\ \operatorname{div}\boldsymbol{v}),\\
&a_\varepsilon(\boldsymbol{u},\ \boldsymbol{v})=a_0(\boldsymbol{u},\ \boldsymbol{v})+\varepsilon^{-1}G(\boldsymbol{u},\ \boldsymbol{v}), \qquad \forall\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\in X. \qquad (2.2)\\
&a_1(\boldsymbol{u};\ \boldsymbol{v},\ \boldsymbol{w})=((\boldsymbol{u}\nabla)\boldsymbol{v},\ \boldsymbol{w})=\sum_{i,j=1}^n\int_\Omega u^j\,\frac{\partial v^i}{\partial x^j}\,w^i\,dx,\\
&a(\boldsymbol{u};\ \boldsymbol{v},\ \boldsymbol{w})=a_1(\boldsymbol{u};\ \boldsymbol{v},\ \boldsymbol{w})+a_1(\boldsymbol{v};\ \boldsymbol{u},\ \boldsymbol{w}),\ \langle\boldsymbol{F},\ \boldsymbol{v}\rangle=\langle\boldsymbol{f},\ \boldsymbol{v}\rangle+\langle\boldsymbol{g},\ \boldsymbol{v}\rangle_{\Gamma_2},
\end{aligned}$$

We assume $\boldsymbol{f}\in V'$ (dual space to V), $\boldsymbol{g}\in[H^{-1/2}(\Gamma_2)]^n$, and $H^{1/2}(\Gamma_2)=$ {the restriction to Γ_2 of $\gamma_0 g$, $g\in H^1(\Omega)$}, γ_0 is the trace mapping from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. If μ is in $H^{1/2}(\Gamma_2)$, we define

$$\|\mu\|_{1/2,\Gamma_2}=\inf_{g\in H^1(\Omega)}\{\|g\|_1,\ \mu=\gamma_0 g|_{\Gamma_2}\}.$$

Let $H^{-1/2}(\Gamma_2)^*$ be the dual space to $H^{1/2}(\Gamma_2)$, normed by

$$\|\mu^*\|_{-1/2,\Gamma_2}=\sup_{\mu\in H^{1/2}(\Gamma_2)}|\langle\mu^*,\ \mu\rangle_{\Gamma_2}|/\|\mu\|_{1/2,\Gamma_2}, \quad \forall\mu^*\in H^{-1/2}(\Gamma_2),$$

where $\langle\cdot,\ \cdot\rangle_{\Gamma_2}$ denotes the duality between $H^{1/2}(\Gamma_2)$ and $H^{-1/2}(\Gamma_2)$. It is not difficult to prove $\langle\boldsymbol{f},\ \boldsymbol{v}\rangle+\langle\boldsymbol{g},\ \boldsymbol{v}\rangle_{\Gamma_2}$, $\forall\boldsymbol{f}\in V'$, $\boldsymbol{g}\in[H^{-1/2}(\Omega)]^n$, to be a linear continuous functional on the space V. Therefore there is an $\boldsymbol{F}\in V'$ such that

$$\langle\boldsymbol{F},\ \boldsymbol{v}\rangle=\langle\boldsymbol{f},\ \boldsymbol{v}\rangle+\langle\boldsymbol{g},\ \boldsymbol{v}\rangle_{\Gamma_2}, \quad \forall\boldsymbol{v}\in V \qquad (2.3)$$

and

$$\|\boldsymbol{F}\|_*\leqslant\|\boldsymbol{f}\|_*+\|\boldsymbol{g}\|_{-1/2,\Gamma_2},$$

where $\|\cdot\|_*$ denotes the dual norm of V'.

In the velocity-pressure formulation, the variational form of (2.1) is

$$\begin{cases}\text{to find } \boldsymbol{u}\in V_0 \text{ such that}\\ a_0(\boldsymbol{u},\ \boldsymbol{v})+a_1(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle, \quad \forall\boldsymbol{v}\in V_0.\end{cases} \qquad (2.4)$$

In the penalty function formulation, the variational form of (2.1) is

$$\begin{cases}\text{to find } \boldsymbol{u}_\varepsilon\in V \text{ such that}\\ a_\varepsilon(\boldsymbol{u},\ \boldsymbol{v})+a_1(\boldsymbol{u}_\varepsilon;\ \boldsymbol{u}_\varepsilon,\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle, \quad \forall\boldsymbol{v}\in V.\end{cases} \qquad (2.5)$$

In both case, the variational form of (2.1) can be written as

$$\begin{cases}\text{to find } \boldsymbol{u}\in H \text{ such that}\\ A(\boldsymbol{u},\ \boldsymbol{v})+a_1(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{v})=\langle\boldsymbol{F},\ \boldsymbol{v}\rangle, \quad \forall\boldsymbol{v}\in H.\end{cases} \qquad (2.6)$$

In the case of (2.4), $H=V_0$ and $A(\boldsymbol{u},\ \boldsymbol{v})=a_0(\boldsymbol{u},\ \boldsymbol{v})$; in the case of (2.5), $H=V$ and $A(\boldsymbol{u},\ \boldsymbol{v})=a_\varepsilon(\boldsymbol{u},\ \boldsymbol{v})$.

We introduce the functional

$$J(\boldsymbol{v})=A(\boldsymbol{v}-\boldsymbol{\xi},\ \boldsymbol{v}-\boldsymbol{\xi})/2, \qquad (2.7)$$

where $\boldsymbol{\xi}$ is a solution of the following Stokes problem:

$$\boldsymbol{\xi}\in H,\ A(\boldsymbol{\xi},\ \boldsymbol{\eta})=\langle\boldsymbol{F},\ \boldsymbol{\eta}\rangle-a_1(\boldsymbol{v};\ \boldsymbol{v},\ \boldsymbol{\eta}), \quad \forall\boldsymbol{\eta}\in H. \qquad (2.8)$$