ESTIMATION OF THE SEPARATION OF TWO MATRICES*

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Abstract

The estimation of the solution to the matrix equation $AX-XB=C$ is primarily dependent on the quantity $\mathrm{sep}(A, B)$ introduced by Stewart[5]. Varah[6] has given some examples to show that $\mathrm{sep}_F(A, B)$ can be very small even though the eigenvalues of A and B are well separated. In this paper we give some lower bounds of $\mathrm{sep}_F(A, B)$.

§ 1. Introduction

Investigations of perturbation bounds for invariant subspaces are frequently reduced to estimations of upper bounds for the solution X to the matrix equation[1,5]

$$AX-XB=C \quad (\lambda(A)\cap\lambda(B)=\emptyset),$$

where $\lambda(\cdot)$ denotes the set of all eigenvalues of a matrix, $\emptyset$ is the empty set. Stewart[5] has difined the separation between A and B

$$\mathrm{sep}(A, B)=\min_{\|X\|=1}\|AX-XB\|, \tag{1.1}$$

where $\|\ \|$ is any matrix norm; thus we obtain

$$\|X\|\leqslant\|C\|/\mathrm{sep}(A, B).$$

Therefore it is necessary to find lower bounds of $\mathrm{sep}(A, B)$ whenever one is investigating perturbations of invariant subspaces.

For A and B normal, Stewart[5] shows that if $\lambda(A)=\{\lambda_i\}$ and $\lambda(B)=\{\mu_i\}$ then

$$\mathrm{sep}_F(A, B)\equiv\min_{\|X\|_F=1}\|AX-XB\|_F=\min_{i,j}|\lambda_i-\mu_j|, \tag{1.2}$$

where $\|\ \|_F$ is the Frobenius norm. However, for A and B non-normal, and

$$\lambda(A)\cap\lambda(B)=\emptyset,$$

up to now we have only the following estimation[5]

$$0<\mathrm{sep}(A, B)\leqslant\min_{i,j}|\lambda_i-\mu_j|; \tag{1.3}$$

and Varah[6] has given some examples to show that $\mathrm{sep}_F(A, B)$ can be very small even though the eigenvalues of A and B are well separated.

In this paper we try to give some lower bounds of $\mathrm{sep}_F(A, B)$. We use reductions of A and B to Jordan canonical forms in § 2 and to some block diagonal forms in § 3.

Notation. The symbol $\mathbf{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices. $I^{(n)}$ is the $n\times n$ identity matrix, and O is the null matrix. Sometimes we express the block

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diagonal matrix $[A_1, \cdots, A_p]$ as $[\cdots, A_i, \cdots]_{(p)}$. $\begin{pmatrix} 0 & [A_1, \cdots, A_s] \\ 0 & 0 \end{pmatrix}_{(p)}$ denotes the matrix $\begin{pmatrix} 0 \cdots 0 & A_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & A_s \\ & & \ddots & 0 \\ & & & \vdots \\ 0 & & & 0 \end{pmatrix}$ in which every row and column contains p submatrices.

Let $\| \ \|_2$ denote the spectral norm and $\kappa(Q) = \|Q\|_2 \|Q^{-1}\|_2$. For $A \in \mathbb{C}^{m\times m}$ with $\lambda(A) = \{\lambda_i\}$ we write $\Delta_F(A) = \{\|A\|_F^2 - \sum_{i=1}^{m} |\lambda_i|^2\}^{\frac{1}{2}}$.

§ 2. Lower bounds of $\mathrm{sep}_F(A, B)$ (I)

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $\lambda(A) \cap \lambda(B) = \emptyset$ and $X \in \mathbb{C}^{m\times n}$. Now we consider to estimate lower bounds of the separation

$$\mathrm{sep}_F(A, B) \equiv \min_{\|X\|_F=1} \|AX - XB\|_F. \tag{2.1}$$

First of all we use the Kronecker product to get another representation for $\mathrm{sep}_F(A, B)$. The Kronecker product of any two matrices $C = (c_{ij}) \in \mathbb{C}^{p\times q}$ and $D \in \mathbb{C}^{r\times s}$ is the matrix $C \otimes D = (c_{ij}D) \in \mathbb{C}^{pr\times qs}$. We associate the matrix X in (2.1) with the mn-vector x which is the direct sum of the column vectors of X. Use the same method we associate the matrix $AX - XB$ in (2.1) with the mn-vector Tx, where

$$T = I^{(n)} \otimes A - B^T \otimes I^{(m)} \in \mathbb{C}^{mn\times mn} \tag{2.2}$$

(see [4], 8—9), B^T stand for transpose of B. From $\lambda(A) \cap \lambda(B) = \emptyset$, the matrix T is nonsingular (see [3], 259. Theorem 8.3.1), thus we obtain

$$\mathrm{sep}_F(A, B) = \min_{\|x\|_2=1} \|Tx\|_2 = \min_{x\neq 0} \frac{\|Tx\|_2}{\|x\|_2} = \left(\max_{y\neq 0} \frac{\|T^{-1}y\|_2}{\|y\|_2}\right)^{-1} = \|T^{-1}\|_2^{-1}. \tag{2.3}$$

Suppose that the Jordan canonical decomposition of A and B^T are

$$A = Q_A J_A Q_A^{-1}, \qquad B^T = Q_B J_B Q_B^{-1}, \tag{2.4}$$

where

$$\begin{gathered} J_A = \Lambda_A + N_A,\ \Lambda_A = [\cdots, \lambda_i I^{(m_i)}, \cdots]_{(p)},\ N_A = [\cdots, N_i(A), \cdots]_{(p)}, \\ N_i(A) = [\cdots, N_{i,k}(A), \cdots]_{(k_i)} \in \mathbb{C}^{m_i\times m_i},\ 1 \leqslant i \leqslant p; \\ J_B = \Lambda_B + N_B,\ \Lambda_B = [\cdots, \mu_j I^{(n_j)}, \cdots]_{(q)},\ N_B = [\cdots, N_j(B), \cdots]_{(q)}, \\ N_j(B) = [\cdots, N_{j,l}(B), \cdots]_{(l_j)} \in \mathbb{C}^{n_j\times n_j},\ 1 \leqslant j \leqslant q. \end{gathered} \tag{2.5}$$

All the matrices $N_{i,k}(A) \in \mathbb{C}^{m_{i,k}\times m_{i,k}}$ and $N_{j,l}(B) \in \mathbb{C}^{n_{j,l}\times n_{j,l}}$ are nilpotent as $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$, $\sum_{k=1}^{k_i} m_{i,k} = m_i$, $\sum_{i=1}^{p} m_i = m$, $\sum_{l=1}^{l_j} n_{j,l} = n_j$ and $\sum_{j=1}^{q} n_j = n$. $\lambda_s \neq \lambda_t$ and $\mu_s \neq \mu_t$ if $s \neq t$. The highest orders of the Jordan blocks of J_A and J_B are r_A and r_B respectively, i.e.

$$r_A = \max_{i,k} \{m_{i,k}\}, \quad r_B = \max_{j,l} \{n_{j,l}\}. \tag{2.6}$$