SEVERAL ABSTRACT ITERATIVE SCHEMES FOR SOLVING THE BIFURCATION AT SIMPLE EIGENVALUES*

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Abstract

In this paper we consider the nonlinear operator equation $\lambda x = Lx + G(\lambda, x)$, where L is a closed linear operator of $X \to X$, X is a real Banach space, with a simple eigen value $\lambda_0 \neq 0$. We discretize its Liapunov–Schmidt bifurcation equation instead of the original nonlinear operator equation and estimate the approximating order of our approximate solution to the genuine solution. Our method is more convenient and more accurate. Meanwhile we put forward several abstract Newton-type iterative schemes, which are more efficient for practical computation, and get the result of their super-linear convergence.

1. Introduction

We consider the nonlinear operator equation in a real Banach space X

$$\lambda x = Lx + G(\lambda, x), \tag{1}$$

where L is a closed linear operator of $X \to X$ with a real simple eigenvalue $\lambda_0 \neq 0$, such that $\lambda_0 I - L$ is a Fredholm operator of index zero, and $G(\lambda, x)$ is a twice continuously differentiable operator of a neighborhood of $(\lambda_0, \theta) \in R \times X \to X$, $G(\lambda, \theta) \equiv \theta$. $G(\lambda, x) = O(\|x\|^2)$, $G_\lambda(\lambda, x) = O(\|x\|^2)$ hold uniformly for λ near λ_0. It is well known that λ_0 is a bifurcation point from the trivial solution of (1) (see [1]). That is, in the neighborhood of (λ_0, θ) there exists $(\lambda(\varepsilon), x(\varepsilon)) \neq (\lambda, \theta)$ which satisfies (1). Moreover

$$\lambda(0) = \lambda_0; \quad x(\varepsilon) = \varepsilon(u_0 + v(\varepsilon)); \quad u_0 \in N(\lambda_0 I - L);$$
$$v(0) = \theta; \quad v(\varepsilon) \in R(\lambda_0 I - L).$$

Here $N(\lambda_0 I - L)$ and $R(\lambda_0 I - L)$ denote the null space and the range of the operator $(\lambda_0 I - L)$ respectively, $\lambda(\varepsilon)$ and $v(\varepsilon)$ are continuously differentiable functions of ε.

In order to compute the bifurcation solution near the simple eigenvalue λ_0, the usual method is first to discretize the original nonlinear operator equation (1), and then to solve the finite dimensional bifurcation problem. Convergence of this method was proved by Atkinson[2], and Weiss[3]. When the eigenvalue λ_0 and its corresponding eigenelement were known in advance, Westreich and Varol[4] proposed an abstract iterative scheme as follows. Let Q be the canonical projection of X onto $N(\lambda_0 I - L)$ and let $Q^* = I - Q$. By means of the Liapunov–Schmidt method, the nonlinear operator equation (1) can be expressed in the equivalent form

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$$\begin{cases} \lambda s v = sLv + Q^* G(\lambda, s(u_0+v)), \\ \lambda s u_0 = sLu_0 + QG(\lambda, s(u_0+v)), \end{cases} \tag{2}$$

where $v \in R(\lambda_0 I - L)$. Let ϕ be the bounded linear functional such that $\phi[u_0]=1$ and $\phi[v]=0$ for $v\in R(\lambda_0 I - L)$. Then $Qx=\phi[x]u_0$ for $x\in X$. For example, if w_0 is an eigenelement of L^* (the adjoint of L) corresponding to λ_0 and w_0 satisfies the normality condition $\langle u_0, w_0\rangle = 1$, $\phi[x]$ can be expressed by $\langle x, w_0\rangle$. Therefore, (1) can also be expressed in the other equivalent form

$$\begin{cases} v = s^{-1}(\lambda I - Q^* L)^{-1} Q^* G(\lambda, s(u_0+v)), \\ \lambda = \lambda_0 + s^{-1}\phi[G(\lambda, s(u_0+v))]. \end{cases} \tag{3}$$

The operator $\lambda I - L$ restricted to $R(\lambda_0 I - L)$ has a uniformly bounded inverse for all λ near λ_0.

The following simple iterative scheme was proposed by Westreich and Varol[4] to solve the bifurcation equation (3)

$$I_1: \quad v^0(s)=\theta, \quad \lambda^0(s)=\lambda_0,$$
$$(\lambda^k(s) I - Q^* L) v^{k+1}(s) = Q^* G(\lambda^k(s), su_0 + v^k(s)),$$
$$\lambda^{k+1}(s) = \lambda_0 + s^{-1}\phi[G(\lambda^k(s), su_0 + v^k(s))].$$

They proved that $\lambda^k(s)$ and $v^k(s)$ converge to the solutions $\lambda(s)$ and $v(s)$ of the bifurcation equation (3) respectively. However, its convergent rate is only linear. In practical numerical computation, the discretization of the original problem is still needed. They pointed out that under some condition the limit of the bifurcation solution of the discretized problem is the solution of the original bifurcation equation (3). However, they did not give the approximating order.

In this paper, we have two purposes. First, we propose several Newton-type iterative schemes for solving bifurcation equations and point out that, their convergent rate is super-linear, they may raise computational efficiency. Second, we directly discretize its Liapunov-Schmidt bifurcation equation (3) instead of the original nonlinear operator equation. Starting from this point of view, we obtain the estimation of the approximating order of our approximate solution to the genuine solution and prove that when λ_0 and u_0 are known in advance, the solution of the discretized bifurcation equation is more accurate and more convenient than the solution of the approximate finite dimensional bifurcation problem.

2. Several Iterative Schemes

In order to simplify our notations, we denote

$$G(\lambda^k(s), s(u_0+v^k(s))) = G^k, \quad \|G^k\| = O(s^2);$$
$$G_\lambda(\lambda^k(s), s(u_0+v^k(s))) = G^k_\lambda, \quad \|G^k_\lambda\| = O(s^2);$$
$$G_u(\lambda^k(s), s(u_0+v^k(s))) = G^k_u, \quad \|G^k_u\| = O(|s|).$$

The equivalent form of the bifurcation equation (3) is

$$\begin{cases} (\lambda I - Q^* L) v = s^{-1} Q^* G(\lambda, s(u_0+v)), \\ \lambda = \lambda_0 + s^{-1}\phi[G(\lambda, s(u_0+v))]. \end{cases} \tag{4}$$

(A) The Newton iterative scheme