NUMERICAL ANALYSIS OF NAVIER-STOKES EQUATIONS BY A HYBRID FINITE ELEMENT METHOD*

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It is known that the advantages of the hybrid finite element method lie in many aspects. Firstly, the trouble of using finite elements of class $C^k (k\geqslant 1)$, which is sometimes required for conforming methods, is avoided. Secondly, some derivatives of the solutions can be obtained simultaneously, such as the stress tensor, which is of more importance sometimes. Thirdly, some special interpolation functions can be easily used for special goals, for instance, we may use the singular expansion as the interpolation functions for fracture mechanics.

The drawback of this method is also obvious, as more variables are involved in the equations to be solved, it is more complicated to construct the stiffness matrix for each element and the program would be more complicated. But the scale of algebraic systems is the same as that of the comforming methods. Therefore if the problem is of large scale and requires a high precision, the hybrid finite element method may be a good choice.

We have applied the hybrid finite element method to incompressible viscous flow[1-3], and discovered another advantage, namely, it improves convergence and stability.

If a primitive variables formulation is used for Navier-Stokes flow, then the Babuska-Brezzi condition is necessary for a conforming finite element approach, the degree of freedom of the velocity field should be much bigger than that of the pressure field, and it causes a loss of precision. For example, as the quadratic six nodes triangular elements are used for the Stokes problem, only a precision of $O(h)$ can be obtained[4], in contrast with the precision $O(h^2)$ for the same elements used in an elastic problem. Some authors have improved the results for Stokes flow, e.g. the work of Santos[5].

We discovered that for the hybrid finite element method, althrough some kinds of Babuska-Brezzi conditions have to be satisfied, they incur no loss of precision. For example, when the quadratic six-nodes triangular elements are used for the velocity field, the precision is $O(h^2)$. Therefore, we get the optimal degree of precision. Some numerical examples have shown that the approximate solution is in good agreement with the analytical solution by our method. In this paper we generalize our method to the nonlinear problem. In the first section we deduce some formulations of the variational problem formally, which means we do not use the terminology of Sobolev spaces, for this kind of statement can give the readers a more intuitional

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understanding of our method. In the second section, we discuss the Stokes problem, which is the foundation of the next section. Most of the material in Section 2 has been published, but we will give a new and simpler proof. In the third section we discuss the hybrid finite element method for Navier–Stokes equations.

§ 1. Some Variational Formulations

Let the fluid be incompressible, viscous, and Newtonian. The space is d-dimensional ($d=2$ or 3), and the governing equations of a stationary flow are

$$-\sigma_{ij,j}+u_j u_{i,j}=f_i, \quad 1\leqslant i\leqslant d, \tag{1.1}$$

$$\sigma_{ij}=\sigma_{ji}=2\nu\varepsilon_{ij}-p\delta_{ij}, \quad 1\leqslant i,\ j\leqslant d, \tag{1.2}$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad 1\leqslant i,\ j\leqslant d, \tag{1.3}$$

$$u_{i,i}=0, \tag{1.4}$$

where $x=(x_i)$ are the spatial cartesian coordinates, f_i are body forces, $\sigma=(\sigma_{ij})$ the stress tensor, which is symmetric, p the hydrostatic pressure, $u=(u_i)$ the velocity, $\varepsilon=(\varepsilon_{ij})$ the velocity strain tensor which is also symmectric, and ν the constant of viscosity. We assume the fluid density is $\rho=1$, and $(\ )_{,i}$ denotes partial differentiation with respect to x_i. For simplification, we will not indicate the range of indices $i,\ j,\ \cdots$.

Let us consider a domain $\Omega\subset R^d$, with boundary $\partial\Omega$. We consider the above equations with boundary value

$$u(x)=\bar{u}_0(x), \quad x\in\partial\Omega, \tag{1.5}$$

where $\bar{u}_0=(\bar{u}_{0i})$ is a known function satisfying

$$\int_{\partial\Omega}\bar{u}_0\cdot n\,dx=0,$$

where $n=(n_i)$ is the unit exterior normal vector along $\partial\Omega$.

First we consider the Stokes equation, that is, the convection term in equation (1.1) is ignored, it becomes

$$-\sigma_{ij,j}=f_i. \tag{1.6}$$

The boundary value problem (1.2)—(1.6) corresponds naturally to the following functional:

$$F_1(\varepsilon,\ u,\ p)=\int_\Omega\{\nu\varepsilon_{ij}(\varepsilon_{ij}-(u_{i,j}+u_{j,i}))+pu_{i,i}+f_iu_i\}dx$$

$$+\int_{\partial\Omega}(2\nu\varepsilon_{ij}-p\delta_{ij})n_j(u_i-\bar{u}_{0i})dx. \tag{1.7}$$

The critical point of (1.7) satisfies equation

$$F_1'(\varepsilon,\ u,\ p)=0, \tag{1.8}$$

that is

$$\int_\Omega\nu\Big(\varepsilon_{ij}-\frac{1}{2}(u_{i,j}+u_{j,i})\Big)\mu_{ij}\,dx=0, \quad \forall\mu_{ij}\ (\mu_{ij}=\mu_{ji}), \tag{1.9}$$

$$\int_\Omega\{-\nu\varepsilon_{ij}(v_{i,j}+v_{j,i})+pv_{i,i}+f_iv_i\}\,dx+\int_{\partial\Omega}(2\nu\varepsilon_{ij}-p\delta_{ij})n_jv_i\,dx=0, \quad \forall v_i, \tag{1.10}$$

$$\int_\Omega u_{i,i}q\,dx=0, \quad \forall q, \tag{1.11}$$