TWO ALGORITHMS FOR SOLVING A KIND OF HEAT CONDUCTION EQUATIONS*

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1. Introduction

In this paper, a strategy is suggested for numerical solution of a kind of parabolic partial differential equations with nonlinear boundary conditions and discontinuous coefficients, which arise from practical engineering problems. First, a difference equation at the discontinuous point is established in which both the stability and the truncation error are consistent with the total difference equations. Then, on account of the fact that the coefficient matrix of the difference equations is tridiagonal and nonlinearity appears only in the first and the last equations, two algorithms are suggested: a mixed method combining the modified Gaussian elimination method with the successive recursion method, and a variant of the modified Gaussian elimination method. These algorithms are shown to be effective.

2. The Difference Equations of the Problem

Consider the parabolic partial differential equation

$$\frac{\partial u}{\partial t}=C\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u, the required solution, satisfies the initial condition

$$u(x,\ 0)=\varphi(x), \tag{2}$$

and the boundary conditions

$$\left[\frac{\partial u}{\partial x}-\lambda_1 u\right]_{x=a}=v_1, \tag{3}$$

$$\left[\frac{\partial u}{\partial x}+\lambda_2 u\right]_{x=b}=v_2, \tag{4}$$

and the discontinuous condition is

$$k_1\frac{\partial u}{\partial x}\Big|_{x_1-0}=k_2\frac{\partial u}{\partial x}\Big|_{x_1+0},\qquad u|_{x_1-0}=u|_{x_1+0}, \tag{5}$$

where

$$\lambda_1=\lambda_1(t,\ u),\qquad \lambda_2=\lambda_2(t,\ u),$$
$$v_1=v_1(t,\ u),\qquad v_2=v_2(t,\ u),$$

and

$$C=\begin{cases}c_1 & a\leqslant x<x_1,\\ c_2 & x_1<x\leqslant b,\end{cases}$$

where c_1, c_2, k_1 and k_2 are constants.

For the region $[a\leqslant x\leqslant b,\ 0\leqslant t\leqslant T]$, there are two intervals in the x-direction. The intervals $[a,\ x_1]$ and $[x_1,\ b]$ are covered by the space increments $h_1=(x_1-a)/$

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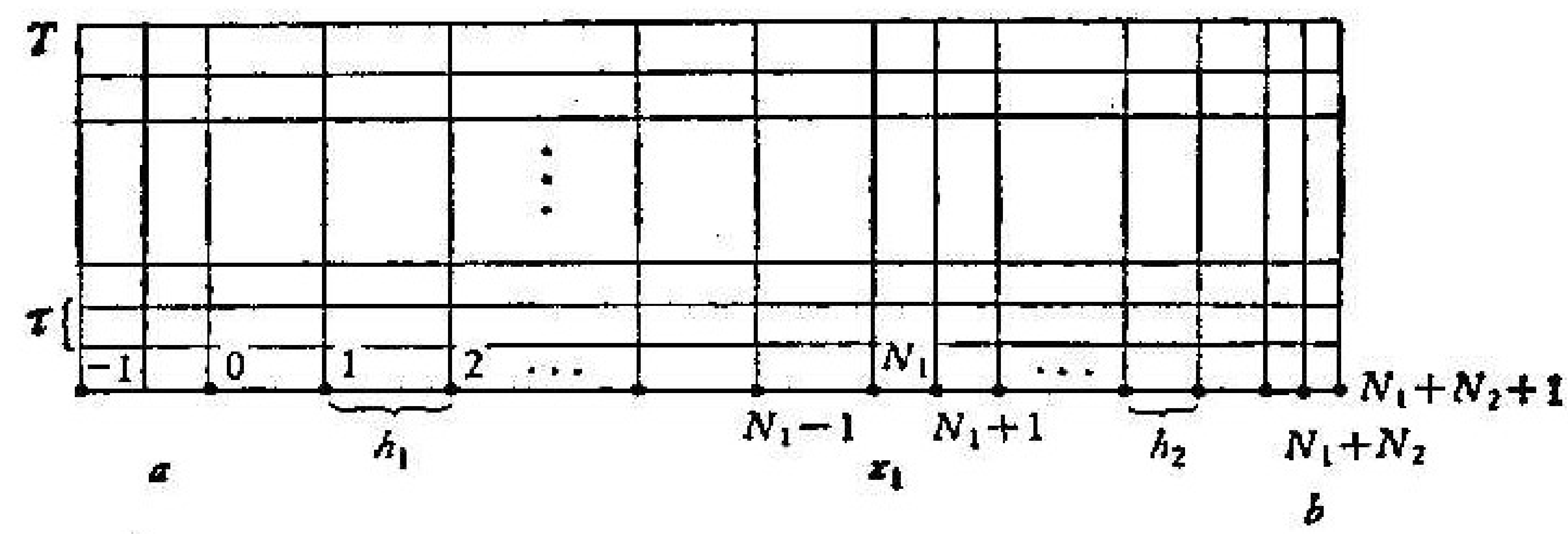
$\left(N_1+\frac{1}{2}\right)$ and $h_2=(b-x_1)\Big/\left(N_2+\frac{1}{2}\right)$ respectively, and time t is covered by $\tau=T/m$. A rectangular net is thus formed as follows.



By use of the Crank-Nicolson formula to approximate equation (1) we obtain the difference equations

$$\begin{gathered}-\gamma_1 u_{k-1,j+1}+(1+2\gamma_1)u_{k,j+1}-\gamma_1 u_{k+1,j+1}\\ =\gamma_1 u_{k-1,j}+(1-2\gamma_1)u_{k,j}+\gamma_1 u_{k+1,j}\\ (k=0,\ 1,\ 2,\ \cdots,\ N_1-1;\ j=1,\ 2,\ \cdots,\ m),\end{gathered}\tag{6}$$

$$\begin{gathered}-\gamma_2 u_{k-1,j+1}+(1+2\gamma_2)u_{k,j+1}-\gamma_2 u_{k+1,j+1}\\ =\gamma_2 u_{k-1,j}+(1-2\gamma_2)u_{k,j}+\gamma_2 u_{k+1,j}\\ (k=N_1+1,\ N_1+2,\ \cdots,\ N_1+N_2;\ j=1,\ 2,\ \cdots,\ m),\end{gathered}\tag{7}$$

where $$\gamma_1=c_1\tau/2h_1^2,\qquad \gamma_2=c_2\tau/2h_2^2.$$

Let $h=\max\{h_1,\ h_2\}$. Obviously equation (1) is approximated by (6) and (7) with the truncation error $O(h^2+\tau^2)$. To obtain the truncation error $O(h^2)$ on the boundary conditions (3) and (4), we replace $\frac{\partial u}{\partial x}$ with the central difference operator, and u with the average of the two adjacent points:

$$\left\{\frac{\partial u}{\partial x}\bigg|_a\right\}_{j+1}=\frac{u_{0,j+1}-u_{-1,j+1}}{h_1},\quad \left\{\frac{\partial u}{\partial x}\bigg|_b\right\}_{j+1}=\frac{u_{N+1,j+1}-u_{N,j+1}}{h_2},$$

$$\{u|_a\}_{j+1}=\frac{u_{0,j+1}+u_{-1,j+1}}{2},\qquad \{u|_b\}_{j+1}=\frac{u_{N+1,j+1}+u_{N,j+1}}{2}.$$

Applying these formulas to (3) and (4) respectively, we have

$$u_{-1,j+1}=\frac{1-\frac{1}{2}h_1\lambda_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}}u_{0,j+1}-\frac{h_1 v_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}},\tag{8}$$

$$u_{N+1,j+1}=\frac{1-\frac{1}{2}h_2\lambda_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}}u_{N,j+1}+\frac{h_2 v_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}},\tag{9}$$

where
$$\begin{aligned}&N=N_1+N_2,\\ &\lambda_{1,j+1}=\lambda_1(t_{j+1},\ (u_{-1,j+1}+u_{0,j+1})/2),\\ &v_{1,j+1}=v_1(t_{j+1},\ (u_{-1,j+1}+u_{0,j+1})/2),\\ &\lambda_{2,j+1}=\lambda_2(t_{j+1},\ (u_{N,j+1}+u_{N+1,j+1})/2),\\ &v_{2,j+1}=v_2(t_{j+1},\ (u_{N,j+1}+u_{N+1,j+1})/2).\end{aligned}$$

Substituting (8) into (6) and (9) into (7), we obtain, for $k=0$ and $k=N$ respec-