IMPLICIT DIFFERENCE SCHEMES FOR THE GENERALIZED NON-LINEAR SCHRÖDINGER SYSTEM*

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Abstract

In this paper we prove under certain weak conditions that two classes of implicit difference schemes for the generalized non-linear Schrödinger system are convergent and that an iteration method for the corresponding non-linear difference equations is convergent. Therefore, quite a complete theoretical foundation of implicit schemes for the generalized non-linear Schrödinger system is established in this paper.

Convergence of Difference Schemes

We discuss the following initial-boundary-value problem for the generalized non-linear Schrödinger system:

$$\begin{cases} iU_t+\dfrac{\partial}{\partial x}\left(A(x)\dfrac{\partial U}{\partial x}\right)+\beta(x)q(|U|^2)U+F(x,\,t)U=G(x,\,t), \\ \qquad\qquad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T, \\ U|_{x=0}=U|_{x=1}=0, \quad 0\leqslant t\leqslant T, \\ U|_{t=0}=E(x), \qquad 0\leqslant x\leqslant 1. \end{cases} \tag{1}$$

Here U, E are complex vectors; $A(x)$, $F(x, t)$ real symmetrical matrices; $\beta(x)$, $q(|U|^2)$ real scalar functions ($|U|$ denotes the Euclidean vector norm of U); and $G(x,t)$ is a real vector. As for $q(|U|^2)$, we consider the following functions: $|U|^2$(and $|U|^{2p}$, p being a positive integer), $\varkappa(1-e^{-|U|^2})$, $|U|^2/(1+|U|^2)$, $\ln(1+|U|^2)$, etc.

This problem can be solved by using the following scheme

$$\begin{cases} i\dfrac{V_j^{n+1}-V_j^n}{\Delta t}+\dfrac{1}{2}\left[(A_{j+\frac12}V_{jx}^{n+1})_{\bar x}+(A_{j+\frac12}V_{jx}^n)_{\bar x}\right]+\dfrac{1}{2}\beta_j(\alpha_1 q(|V_j^{n+1}|^2) \\ \quad +(1-\alpha_1)q(|V_j^n|^2))\cdot(V_j^{n+1}+V_j^n)+\dfrac{1}{2}F_j^{n+\frac12}(V_j^{n+1}+V_j^n)=G_j^{n+\frac12}, \\ \qquad\qquad j=1,\,2,\,\cdots,\,J-1, \\ V_0^{n+1}=V_J^{n+1}=0, \quad J=1/\Delta x. \end{cases} \tag{2}$$

Here V_j^n denotes the approximate value of U at $x=j\Delta x$, $t=n\Delta t$; $F_j^{n+\frac12}=F\left(j\Delta x,\left(n+\frac12\right)\Delta t\right)$; $A_{j+\frac12}=A\left(\left(j+\frac12\right)\Delta x\right)$, $\beta_j=\beta(j\Delta x)$, $G_j^{n+\frac12}=G\left(j\Delta x,\left(n+\frac12\right)\Delta t\right)$; $V_{jx}^{n+\delta}=\dfrac{1}{\Delta x}(V_{j+1}^{n+\delta}-V_j^{n+\delta})$, $(A_{j+\frac12}V_{jx}^{n+\delta})_{\bar x}=\dfrac{1}{\Delta x}(A_{j+\frac12}V_{jx}^{n+\delta}-A_{j-\frac12}V_{j-1,x}^{n+\delta})$, $\delta=0$ or 1; and α_1 is a

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positive constant. Clearly, the truncation error is $O(\Delta t^2+\Delta x^2)$ for $\alpha_1=\frac{1}{2}$, and $O(\Delta t+\Delta x^2)$ otherwise. Chang[1] has discussed convergence of this scheme for $\alpha_1=1$. However, one meets considerable difficulties when trying to prove the convergence of this scheme for $\alpha_1\neq 1$ by using the method in [1]. If $\alpha_1\neq 0$, one has to solve a system of non-linear equations at each step. An iteration method for solving the system is usually needed, and the iteration is required to be convergent. For the scheme with $\alpha_1=0$, only a system of linear equations needs to be solved, so this scheme is also special. Therefore, we pay our attention to the three schemes with $\alpha_1=\frac{1}{2}$, 1, 0, which are called Scheme A, Scheme B and Scheme C respectively in the following.

We first discuss the stability of Scheme A. Clearly, there are the following relations:

$$\left\{\begin{aligned}
&\frac{i}{\Delta t}\sum_{j=1}^{J-1}[(V_j^{n+1}-V_j^n,\ V_j^{n+1}+V_j^n)+(\bar V_j^{n+1}-\bar V_j^n,\ \bar V_j^{n+1}+\bar V_j^n)]\\
&\quad=\frac{2i}{\Delta t}\sum_{j=1}^{J-1}(|V_j^{n+1}|^2-|V_j^n|^2);\\
&\frac{1}{2}\sum_{j=1}^{J-1}[((A_{j+\frac12}V_{jx}^{n+1})_{\bar x}+(A_{j+\frac12}V_{jx}^n)_{\bar x},\ V_j^{n+1}+V_j^n)\\
&\qquad-((A_{j+\frac12}\bar V_{jx}^{n+1})_{\bar x}+(A_{j+\frac12}\bar V_{jx}^n)_{\bar x},\bar V_j^{n+1}+\bar V_j^n)]\\
&\quad=-\frac{1}{2}\sum_{j=0}^{J-1}[(A_{j+\frac12}V_{jx}^{n+1}+A_{j+\frac12}V_{jx}^n,\ V_{jx}^{n+1}+V_{jx}^n)\\
&\qquad-(A_{j+\frac12}\bar V_{jx}^{n+1}+A_{j+\frac12}\bar V_{jx}^n,\ \bar V_{jx}^{n+1}+\bar V_{jx}^n)]=0;\\
&\sum_{j=1}^{J-1}\Big[\frac{1}{4}\beta_j(q(|V_j^{n+1}|^2)+q(|V_j^n|^2))[(V_j^{n+1}+V_j^n,\ V_j^{n+1}+V_j^n)\\
&\qquad-(\bar V_j^{n+1}+\bar V_j^n,\ \bar V_j^{n+1}+\bar V_j^n)]+\Big(\frac{1}{2}F_j^{n+\frac12}(V_j^{n+1}+V_j^n),\ V_j^{n+1}+V_j^n\Big)\\
&\qquad-\Big(\frac{1}{2}F_j^{n+\frac12}(\bar V_j^{n+1}+\bar V_j^n),\ \bar V_j^{n+1}+\bar V_j^n\Big)\Big]=0;\\
&\sum_{j=1}^{J-1}[(G_j^{n+\frac12},\ V_j^{n+1}+V_j^n)-(G_j^{n+\frac12},\ \bar V_j^{n+1}+\bar V_j^n)]\\
&\quad=2i\sum_{j=1}^{J-1}(G_j^{n+\frac12},\ \mathrm{Im}(V_j^{n+1}+V_j^n)).
\end{aligned}\right.\tag{3}$$

Therefore, subtracting the inner product of $\bar V_j^{n+1}+\bar V_j^n$ and the conjugate equation of (2) from that of $V_j^{n+1}+V_j^n$ and (2), and summing up these differences from $j=1$ to $J-1$, we obtain

$$\frac{1}{\Delta t}\sum_{j=1}^{J-1}(|V_j^{n+1}|^2-|V_j^n|^2)=\sum_{j=1}^{J-1}(G_j^{n+\frac12},\ \mathrm{Im}(V_j^{n+1}+V_j^n)).$$

Let $\|V^n\|^2=\Delta x\sum_{j=1}^{J-1}|V_j^n|^2$, $\|G^n\|^2=\Delta x\sum_{j=1}^{J-1}|G_j^n|^2$. Then it follows from the above relation and the Schwarz inequality that

$$\|V^{n+1}\|^2-\|V^n\|^2\leqslant\Delta t\Big(\frac{1}{2}\|V^{n+1}\|^2+\frac{1}{2}\|V^n\|^2+\|G^{n+\frac12}\|^2\Big),$$

which can be rewritten as