THE CONVERGENCE OF INFINITE ELEMENT METHOD FOR THE NON-SIMILAR CASE*

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We have considered the infinite element method for a class of elliptic systems with constant coefficients in [1]. This class can be characterized as: they have the invariance under similarity transformations of independent variables. For example, the Laplace equation and the system of plane elastic equations have this property. We have suggested a technique to solve these problems by applying this property and a self similar discretization, and proved the convergence. Not only the average convergence of the solutions has been discussed, but also the term-by-term convergence for the expansions of the solutions. The second convergence manifests the advantage of the infinite element method, that is, the local singularity of the solutions can be calculated with high precision.

We have generalized this method to the non-similar case in [2], and obtained many results parallel to that of the similar case, which include the calculation of the combined stiffness matrices and the discussion of the singularity of the solutions. For conciseness, the Helmholtz equation

$$-\Delta u+\lambda u=0 \tag{1}$$

and the linear triangular elements will be considered in this paper, but this method is good for more general equations.

We will prove the average convergence which shows the order of the convergence of the infinite element method is higher than that of the finite element method if the solutions possess singularities. At the same time, we will concentrate upon the proof of the convergence for singular components, which does not exist for the finite element method.

§ 1. Some General Statement

For conciseness we assume that the considered region D is a bounded polygon region on the x, y plane, one of the vertices of which is the origin, the inner angle of which is $\theta_0>\pi$. If we consider the boundary value problem of equation (1) on this region, the solution will, generally, possess singularity at point O. Now we devote ourselves to the calculation of this singularity. It is no harm to assume that one of the neighboring sides of point O is on the positive x-axis and the interior angle is $0<\theta<\theta_0$.

Let the neighboring sides of point O be Γ' and Γ'', we construct a neighborhood Ω_0 of point O in D, which is a polygon region and satisfies the star-shape condition with

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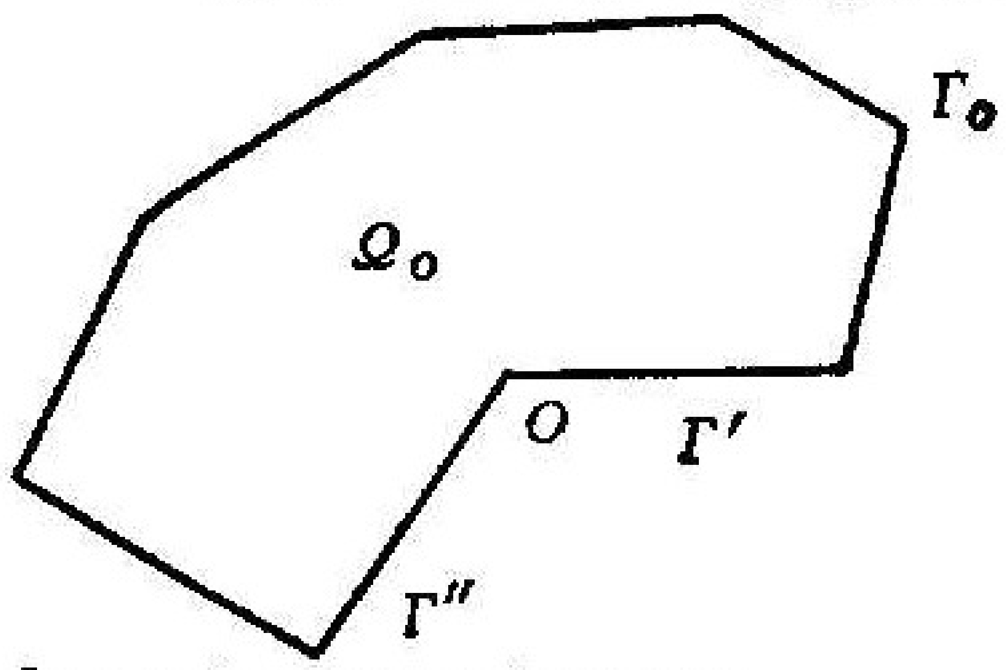
respect to point O, that is the line segment which connects any point on $\bar{\Omega}_0$ and point O lies on $\bar{\Omega}_0$ entirely. Γ' and Γ'' are also two sides of Ω_0, the remaining part of the boundary of Ω_0 is expressed by $\Gamma_0: r=R(\theta)$, where r, θ are polar coordinates.



Some boundary conditions are assumed on the boundary of D, for definiteness we assume that the homogeneous Neumann condition $\frac{\partial u}{\partial \nu}=0$ is assumed on Γ'andΓ'', where ν denotes the normal directoin, the discussion is similar for other kind of homogeneous boundary conditions.

We make the infinite element discretization as usual: $D\setminus\bar{\Omega}_0$ is discretized into a finite number of triangular elements by the conventional way, while Ω_0 is discretized into an infinite number of triangular elements as the following: a constant $\xi, 0<\xi<1$, is taken, we construct similar curves of Γ_0 with 0 as the center and $\xi, \xi^2, \cdots, \xi^k, \cdots$ as the constants of proportionality, thus, Ω_0 is discretized into an infinite number of "layers", then we construct the line segments from point O to nodal points on Γ_0, each layer is discretized into finite number of quadrilaterals, finally, each quadrilateral is discretized into triangles, the mode of discretization for each layer is the same. The interpolation functions are linear on each element and is continuous on D.

One auxilary unbounded region is needed in the following proof, which is denoted by $\Omega=\{(r, \theta): 0<r<\infty, 0<\theta<\theta_0\}$, hence $\Omega_0\subset\Omega$. We also discretize region $\Omega\setminus\bar{\Omega}_0$ into infinite number of layers by the constants of proportionality $\xi^{-1}, \cdots, \xi^{-k}, \cdots$, then they are discretized into triangular elements by the same mode. The curves $r=\xi^k R(\theta)$ $(k=0, \pm1, \cdots)$are denoted by Γ_k.

The region Ω_0 with its discretization is called a combined element, since equation (1) is given, if u_h is an approximate solution by the infinite element method[2], then the "strain energy"associated with u_h on Ω_0,

$$\int_{\Omega_0}|\nabla u_h|^2 dx\, dy+\lambda\int_{\Omega_0}|u_h|^2 dx\, dy$$

would be determined by the values of u_h at the nodes of Γ_0, where ∇ is the gradient operator: $\nabla:\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. Suppose there are m nodes on Γ_0, the values of u_h on which form a m-dimensional vector according to a definite order, say the anti-clockwise order, which is denoted by y_0. A technique in [2] is given to calculate the combined stiffness matrix $K_z(\lambda)$, then for any boundary value y_0, the strain energy of the approximate solution u_h associated with y_0 on Ω_0 is expressed by $\frac{1}{2}\bar{y}_0^T K_z(\lambda) y_0$, where T denotes transpose. Ω_0 can be treated as one element by means of the matrix $K_z(\lambda)$, we can solve this algebraic system as ordinary finite element method by assembling this combined element with conventional elements on $D\setminus\bar{\Omega}_0$. We denote the values of u_h at the nodes of Γ_k by vectors y_k $(k=1, 2, \cdots)$ just as y_0. When we consider the approximate solutions on Ω, we also use the symbols y_k for negative indices k.

We assume that the above discretization is normal, that is, any two closed triangular elements possess either a common vertex or a common side, or no common point at all. We assume also all the inner angles of elements have an upper bound