THE ERROR BOUND OF THE FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL SINGULAR BOUNDARY VALUE PROBLEM*

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1. Introduction

The finite element method for one-dimensional singular boundary value problems have been studied by several authors (for instance, see [4], [10], [8], [11]). The finite element method for a two-dimensional singular boundary value problem is proposed in [12]. Recently [9], [16], [1], [15] and [3] have given the relevant theoretical studies. In [9], the error of order $O(h^k)$ has been proved for the Lagrange elements of degree k provided that the solution of the boundary value problem is in $C^{k+1}(\bar{\Omega})$. [16] has proved the convergence of the linear finite element method provided only that the solution of the boundary value problem belongs to a weighted Sobolev space. For problem (1.1) in the present paper, [1] has proved that the error is of order $O(h)$ for a variant linear element including a logarithmic term. For the ordinary linear element, [15] and [3] have also obtained the error of order $O(h)$. In this paper we extend the result of [15] and [3] to the elements of high degree.

We consider the following model problem:

$$\begin{cases} \Omega: & Lu \equiv -\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\beta_1\dfrac{\partial u}{\partial r}\right)+\dfrac{\partial}{\partial z}\left(\beta_2\dfrac{\partial u}{\partial z}\right)\right]=f, \\ \Gamma_1: & u=0, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain with $r>0$ in (r, z)-plane, $\Gamma_1=\partial\Omega\backslash\Gamma_0$, $\Gamma_0=\partial\Omega\cap\{(r, z): r=0\}$.

In order to formulate the weak from of problem (1.1) we introduce some weighted Sobolev spaces. The similar spaces have been studied in [2], [5], [13] and [14].

2. Weighted Sobolev Spaces V_1^m

Define
$$V^0(\Omega)=\{v: v \text{ is measurable in } \Omega,\ \|v\|_{V^0(\Omega)}<\infty\},$$
$$V_1^m(\Omega)=\{v\in V^0(\Omega): \|v\|_{V_1^m(\Omega)}<\infty\},\ m=1, 2, \cdots,$$
where
$$\|v\|_{V^0(\Omega)}=\left(\int_\Omega v^2 r\,dr\,dz\right)^{1/2},$$
$$\|v\|_{V_1^1(\Omega)}=\Big(\sum_{|\alpha|\le 1}\|\partial^\alpha v\|^2_{V^0(\Omega)}\Big)^{1/2},$$

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$$\|v\|_{V_1^m(\Omega)}=\Big(\sum_{|\alpha|<m}\|\partial^\alpha v\|^2_{V^0(\Omega)}+\sum_{j=1}^{m-1}\Big\|r^{j-m}\frac{\partial^j v}{\partial r^j}\Big\|^2_{V^0(\Omega)}\Big)^{1/2},\quad m=2,\ 3,\ \cdots.$$

Sometimes we use V^0, V_1^m instead of $V^0(\Omega)$, $V_1^m(\Omega)$.

Using the arguments similar to those in [13], [14] and [5] we can prove the following propositions.

Proposition 2.1. The spaces V^0, V_1^m are Banach spaces.

Proposition 2.2. If Ω has a locally Lipschitz boundary then $C^\infty(\overline{\Omega})$ is dense in $V_1^m(\Omega)$.

Now we may as usual define the trace on the boundary of Ω for the elements of $V_1^m(\Omega)$. Then we may introduce the following spaces corresponding to problem (1.1):

$$V_{1,0}^1(\Omega)=\{v\in V_1^1(\Omega):\ v=0 \text{ on } \Gamma_1\}.$$

From now on we assume that Ω has a locally Lipschitz boundary, that $f\in V^0(\Omega)$, and that β_1, β_2 are bounded, measurable in Ω and there exists a positive constant β_0 such that $\beta_1\geqslant\beta_0$, $\beta_2\geqslant\beta_0$.

Lemma 2.3. (Ref. [6]) *There exists a constant $C>0$ such that*

$$\int_\Omega\Big[\Big(\frac{\partial v}{\partial r}\Big)^2+\Big(\frac{\partial v}{\partial z}\Big)^2\Big]r\,dr\,dz\geqslant C\|v\|^2_{V_1^1(\Omega)},\quad \forall v\in V_{1,0}^1(\Omega).$$

The proof of the following lemma is similar to that of theorem 2.2 in [5].

Lemma 2.4. *If $v\in V_1^m$, $m\geqslant 2$, then*

$$\frac{\partial^j v}{\partial r^j}=0 \text{ on } \Gamma_0,\qquad j=1,\ 2,\ \cdots,\ m-1.$$

It is easy to prove that $V_1^2(\Omega)\subset C^0(\overline{\Omega})$. (Ref. [15]).

3. The Weak Form of the Problem and the Discrete Problem

Define the bilinear form $B_1(u,v)$ and the linear functional $F(v)$ as follows:

$$B_1(u,v)=\int_\Omega\Big(\beta_1\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\beta_2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\Big)r\,dr\,dz,\qquad \forall u,\ v\in V_1^1(\Omega),$$

$$F(v)=\int_\Omega fvr\,dr\,dz,\qquad \forall v\in V_1^1(\Omega).$$

The weak form of problem (1.1) is

Problem (3.1). Find $u\in V_{1,0}^1(\Omega)$ such that

$$B_1(u,v)=F(v),\ \forall v\in V_{1,0}^1(\Omega).$$

By lemma 2.3 we know that $B_1(u,v)$ is coercive on $V_{1,0}^1(\Omega)\times V_{1,0}^1(\Omega)$. So we may easily prove the following theorem using the Lax-Milgram theorem.

Theorem 3.1. *Problem (3.1) has a unique solution.*

From now on we assume that Ω is a polygon.

Let $T_h=\{C_1,\ \cdots,\ C_n\}$ be a normal triangulation of Ω (Ref. [6]). Denote by h_i and θ_i respectively the size of the maximal edge and the minimal inner angle of C_i. Let $h=\max h_i$, $\theta=\min\theta_i$. Define the finite element spaces $V_1^{m,h}$ of degree m as follows:

$$V_1^{1,h}=\{v_h\in C^0(\overline{\Omega}):\ v_h \text{ is a linear function on } C_i, i=1,\ \cdots,\ n;\ v_h=0 \text{ on } \Gamma_1\},$$

$$V_1^{m,h}=\{v_h\in C^{m-1}(\overline{\Omega}):\ v_h \text{ is a polynomial of degree } m \text{ on } C_i,\ i=1,\ \cdots,\ n;\ v_h=0 \text{ on } \Gamma_1\},\ m=2,\ 3,\ \cdots.$$