DIFFERENCE SCHEMES OF DEGENERATE PARABOLIC EQUATIONS*

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§1

Consider the partial differential equation of second order

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\sigma\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial x}+du+f, \qquad (x, t)\in I\times(0, T], \tag{1}$$

where the unknown function u and coefficients σ, b, d, f are functions of x and t. Denote the interval $0<x<l$ by I. Let $Z\subset\bar{I}\times(0, T]$ be the point set, on which $\sigma=0$. If $\sigma(x, t)\geqslant 0$ on the domain $\bar{I}\times(0, T]$, and Z is not an empty set, then equation (1) is known as a degenerate parabolic equation. In order for the initial and boundary-value problem of the equation (1) to be properly posed, the initial and boundary conditions must be appropriate. The boundary conditions to be posed depend on the behaviour of coefficients $\sigma(x, t)$ and $b(x, t)$ on $x=0$ and $x=l$. If $\sigma(0,t)=0$, $b(0,t)<0$ simultaneously, or $\sigma(0, t)>0$, then on $x=0$, a boundary condition should be given; otherwise (i. e. if $\sigma(0, t)=0$ and $b(0, t)\geqslant 0$ simultaneously), no boundary condition on $x=0$ is needed. On $x=l$, when $\sigma(l, t)=0$, $b(l, t)>0$ simultaneously, or $\sigma(l, t)>0$, a boundary condition should be given; otherwise, it is not needed. Moreover, for equation (1), the initial condition

$$u(x, 0)=g_0(x), \qquad x\in\bar{I} \tag{2}$$

is always needed[1].

In this section we suppose

$$\sigma(0, t)=\sigma(l, t)=0,\ b(0, t)\geqslant 0,\ b(l, t)\leqslant 0,\ t\in(0, T]. \tag{3}$$

In addition, we assume that the coefficients of the equation (1) are sufficiently smooth and that there exists a unique sufficiently smooth solution of the equation (1) with initial condition (2).

We solve the problem (1), (2) by a difference method. Divide the interval $[0, l]$ and $[0, T]$ into J and N parts respectively. The space step is $h=l/J$ and the time step is $\tau=T/N$. Let $\omega_h=\{x_j=jh\,|\,j=0, 1, \cdots, J\}$ and $\omega_\tau=\{t^n=n\tau\,|\,n=0, 1, \cdots, N\}$. The set of all net points on the domain $\bar{I}\times[0, T]$ is denoted by $\omega_h\times\omega_\tau$.

Let $y(x, t)$ and $z(x, t)$ be functions, defined on the set $\omega_h\times\omega_\tau$. Introduce the following notations

$$y_j^n=y(jh, n\tau)$$

$$y_{x,j}^n=\frac{1}{h}(y_{j+1}^n-y_j^n), \qquad y_{\bar{x},j}^n=\frac{1}{h}(y_j^n-y_{j-1}^n).$$

Define the inner products

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$$(y^n, z^n) = \sum_{j=1}^{J-1} \alpha_j y_j^n z_j^n h, \qquad [y^n, z^n) = \sum_{j=0}^{J-1} \alpha_j y_j^n z_j^n h,$$

$$(y^n, z^n] = \sum_{j=1}^{J} \alpha_j y_j^n z_j^n h, \qquad [y^n, z^n] = \sum_{j=0}^{J} \alpha_j y_j^n z_j^n h,$$

where $\alpha_0 = \alpha_J = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{J-1} = 1$, and consequently, the norms

$$\|y^n\| = (y^n, y^n)^{\frac{1}{2}}, \qquad |[y^n\| = [y^n, y^n)^{\frac{1}{2}},$$
$$\|y^n]| = (y^n, y^n]^{\frac{1}{2}}, \qquad |[y^n]| = [y^n, y^n]^{\frac{1}{2}}. \tag{4}$$

Because the boundary condition is given neither on $x=0$ nor on $x=l$, the difference scheme should be constructed on each point of the set $\omega_h \times (\omega_\tau \backslash t^0)$. On $x=0$ and $x=l$, the equation (1) can be reduced to the following form

$$\frac{\partial u}{\partial t} = (\sigma' + b)\frac{\partial u}{\partial x} + du + f,$$

where σ' denotes $\partial\sigma/\partial x$. Since the function $\sigma(x, t)$ is non-negative on the whole domain $\bar{I} \times (0, T]$, it is clear that

$$\sigma'(0, t) \geqslant 0, \quad \sigma'(l, t) \leqslant 0. \tag{5}$$

Let $y(x, t)$ be a function defined on the set $\omega_h \times \omega_\tau$. The Crank-Nicholson scheme approximating the differential equation (1) is

$$\frac{y_0^{n+1} - y_0^n}{\tau} = (\sigma_0'^{\,n+\frac{1}{2}} + b_0^{n+\frac{1}{2}})\frac{1}{2}(y_{x,0}^{n+1} + y_{x,0}^n) + d_0^{n+\frac{1}{2}}\frac{1}{2}(y_0^{n+1} + y_0^n) + f_0^{n+\frac{1}{2}}, \tag{6}$$

$$\frac{y_j^{n+1} - y_j^n}{\tau} = \left(a^{n+\frac{1}{2}}\frac{1}{2}(y_{\bar{x}}^{n+1} + y_{\bar{x}}^n)\right)_{x,j} + b_j^{n+\frac{1}{2}}\frac{1}{4}(y_{\bar{x},j}^{n+1} + y_{\bar{x},j}^n + y_{x,j}^{n+1} + y_{x,j}^n)$$
$$+ d_j^{n+\frac{1}{2}}\frac{1}{2}(y_j^{n+1} + y_j^n) + f_j^{n+\frac{1}{2}}, \quad j = 1, 2, \cdots, J-1, \tag{7}$$

$$\frac{y_J^{n+1} - y_J^n}{\tau} = (\sigma_J'^{\,n+\frac{1}{2}} + b_J^{n+\frac{1}{2}})\frac{1}{2}(y_{\bar{x},J}^{n+1} + y_{\bar{x},J}^n) + d_J^{n+\frac{1}{2}}\frac{1}{2}(y_J^{n+1} + y_J^n) + f_J^{n+\frac{1}{2}}, \tag{8}$$

where $a_j^{n+\frac{1}{2}} = \sigma_{j-\frac{1}{2}}^{n+\frac{1}{2}}$, and for any function $\phi(x, t)$, we have $\phi_\alpha^\beta = \phi(\alpha h, \beta\tau)$. The initial condition (2) is approximated by

$$y_j^0 = g_0(jh), \quad j = 0, 1, \cdots, J. \tag{9}$$

Equations (6)—(8) are the system of linear equations with unknowns $y_0^{n+1}, y_1^{n+1}, \cdots, y_J^{n+1}$. Let $C_d = \sup_{I\times[0,T]} \frac{1}{2}(d + |d|)$. When $C_d \neq 0$, the coefficient matrix of equation (6)—(8) is diagonally dominant for $\tau < 2/C_d$, and when $C_d = 0$, for arbitrary τ. Then, the system of difference equations (6)—(8) is solvable.

Let $z(x, t)$ be the difference between the solution of difference equations (6)—(9) and that of the differential equation (1) with initial condition (2), i. e.

$$z(x, t) = y(x, t) - u(x, t), \quad (x, t) \in \omega_h \times \omega_\tau. \tag{10}$$

Putting $y = z + u$ in the difference equations (6)—(9), we obtain

$$\frac{z_0^{n+1} - z_0^n}{\tau} - (\sigma_0'^{\,n+\frac{1}{2}} + b_0^{n+\frac{1}{2}})\frac{1}{2}(z_{x,0}^{n+1} + z_{x,0}^n) - d_0^{n+\frac{1}{2}}\frac{1}{2}(z_0^{n+1} + z_0^n) = \psi_0^{n+\frac{1}{2}}, \tag{11}$$

$$\frac{z_j^{n+1} - z_j^n}{\tau} - \left(a^{n+\frac{1}{2}}\frac{1}{2}(z_{\bar{x}}^{n+1} + z_{\bar{x}}^n)\right)_{x,j} - b_j^{n+\frac{1}{2}}\frac{1}{4}(z_{x,j}^{n+1} + z_{x,j}^n + z_{\bar{x},j}^{n+1} + z_{\bar{x},j}^n)$$