ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE*

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Abstract

In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in $L_2(\Delta)$. The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.

1. Introduction

It is interesting and useful to study multivariate Haar and Walsh functions either in theory or in practice. If we investigate on a domain which can be considered a Cartesian product, then the functions are readily extended to several variables from one variable. Setting by the tensor product construct Harmuth has shown those kinds of multivariate systems in his book[5] and pointed out the applications in communication.

In this paper we attempt to focus on a triangle, or more generally on a simplex in n-dimensional space. We did not find any paper about it. Perhaps it puzzles some people temporarily.

The main contribution of this paper is to establish the Haar and Walsh system with two variables on a triangle. We prove their orthonormality and completeness in Hilbert space L_2. Moreover, the corresponding Haar-Fourier series and Walsh-Fourier series for any continuous function are uniformly convergent and convergent by group respectively.

It is easy to generalize these results to the n-dimensional simplex. For simplicity we will discuss only the two-dimensional triangle.

Now we explain some preliminaries and notations.

The Haar functions on [0, 1] are defined as follows:

$$\chi_0(t):=1, \qquad \text{for } 0\leqslant t\leqslant 1,$$

and

$$\chi_n^{(k)}(t):=\begin{cases}\sqrt{2^n}, & \text{for } \dfrac{2k-2}{2^{n+1}}\leqslant t<\dfrac{2k-1}{2^{n+1}},\\ -\sqrt{2^n}, & \text{for } \dfrac{2k-1}{2^{n+1}}<t\leqslant \dfrac{2k}{2^{n+1}},\\ 0, & \text{elsewhere in } [0,1],\end{cases} \tag{1.1}$$

$$k=1,2,3,\cdots,2^n; \qquad n=1,2,3,\cdots,\infty.$$

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The Walsh functions on [0, 1] consist of the following ones:

$$w_0(t) := 1, \qquad \text{for } 0\leqslant t\leqslant 1,$$

$$w_1(t) := \begin{cases} 1, & \text{for } 0\leqslant t<\frac{1}{2}, \\ -1, & \text{for } \frac{1}{2}<t\leqslant 1, \end{cases} \tag{1.2}$$

$$w_{n+1}^{(2k-1)}(t) := \begin{cases} w_n^{(k)}(2t), & \text{for } 0\leqslant t<\frac{1}{2}, \\ (-1)^{k+1}w_n^{(k)}(2t-1), & \text{for } \frac{1}{2}<t\leqslant 1, \end{cases}$$

$$w_{n+1}^{(2k)}(t) := \begin{cases} w_n^{(k)}(2t), & \text{for } 0\leqslant t<\frac{1}{2}, \\ (-1)^{k}w_n^{(k)}(2t-1), & \text{for } \frac{1}{2}<t\leqslant 1, \end{cases}$$

$$k=1,\ 2,\ 3,\ \cdots,\ 2^n; \qquad n=1,\ 2,\ 3,\ \cdots,\ \infty.$$

Some detailed investigation of the Haar and Walsh systems can be found in [1], [3], [5].

In order to generalize the Haar and Walsh systems to the two-dimensional case we should explain our representation in this paper. The Cartesian coordinates are not very convenient for triangular elements, and a special type of coordinate system called area coordinates should be used.

In Figure 1 it is seen that the internal point P will divide the triangle ABC into three smaller triangles, and depending on the position of the point P, the area of each of the triangles PAB, PBC, PCA can vary from zero to $|\Delta|$, which is the area of the triangle ABC. In other words, the ratios $\frac{a}{|\Delta|}$, $\frac{b}{|\Delta|}$ and $\frac{c}{|\Delta|}$ will take any value between zero and unity. Here a, b, c are the area of triangles PBC, PCA, PAB respectively.

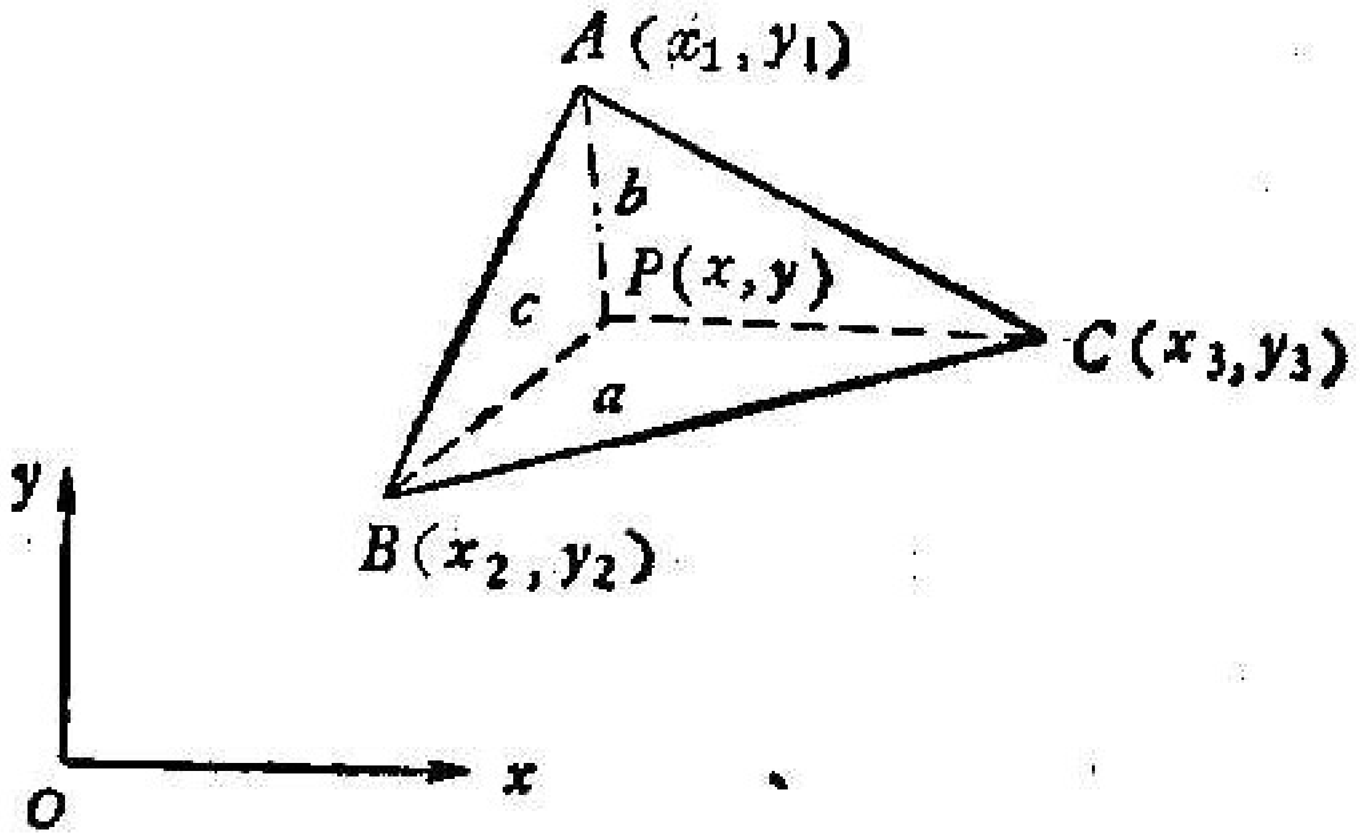

Fig. 1

These ratios are called area coordinates, defined by $l_1 := \frac{a}{|\Delta|}$, $l_2 := \frac{b}{|\Delta|}$, $l_3 := \frac{c}{|\Delta|}$.

It is easy to see that

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}.$$

If two points P and Q are in two similar triangles respectively, and have the same area coordinates, then we denote them by $P\sim Q$.

2. An Orthonormal Sequence χ on a Triangular Domain

Suppose Δ(or Δ_{ABC}) is any triangle on a plane and $|\Delta| = 1$ is the area of Δ_{ABC}. If D, E, F are midpoints of AB, BC, CA respectively, connecting DE, EF, FD, we divide