PERTURBATION THEOREMS FOR GENERALIZED SINGULAR VALUES *1)

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Abstract

Let A and B be $m\times n$ and $p\times n$ complex matrices respectively. This paper, as a continuation of the author's papers [7] (*Math. Numer. Sinica*, 4(1982), 229—233) and [8] (*SIAM J. Numer. Anal.*, to appear), discusses perturbation bounds for the generalized singular values of the matrix-pair $\{A, B\}$ in the case of rank $\begin{pmatrix} A \\ B \end{pmatrix}<n$.

Let m, p and n be arbitrary natural numbers, A and B be $m\times n$ and $p\times n$ complex matrices respectively. Van Loan[10], Paige and Saunders[4] have suggested forms of the generalized singular value decomposition (GSVD) of the matrix-pair $\{A, B\}$. In two later papers[7,8] the author has analysed the perturbation of the singular values and the singular subspaces of $\{A, B\}$ in the case of rank$\begin{pmatrix} A \\ B \end{pmatrix}=n$. In this paper we investigate the perturbation of the singular values of $\{A, B\}$ in the case of rank $\begin{pmatrix} A \\ B \end{pmatrix}<n$ (Perturbation bounds for generalized singular subspaces of $\{A, B\}$ in this case have been given by the author in "The $\sin\theta$ theorems for generalized singular subspaces").

It is well-known that the singular values of an $m\times n$ matrix A are the non-negative square roots of the n eigenvalues of the positive semi-definite matrix A^HA (A^H is the conjugate transpose of A). In § 1 we generalize the singular value concept and derive the GSVD exactly from this point of view. Formerly, any pair (α, β) with $\alpha, \beta\geqslant 0$ and $(\alpha, \beta)\neq(0, 0)$ was regarded as a singular value of $\{A, B\}$ in the case of rank $\begin{pmatrix} A \\ B \end{pmatrix}<n$ (Ref. [10], [4], [7]), and consequently it is difficult to investigatethe perturbation of singular values in this case; we shall clarify this problem in § 1. In § 2 and § 3 we prove a Weyl type theorem and a Hoffman-Wielandt type theorem respectively. The results show that, in the case where $\begin{pmatrix} A \\ B \end{pmatrix}$ is acutely perturbed, if we use the chordal metric to describe the perturbation of singular values, then the singular values of $\{A, B\}$ are insensitive to perturbations in the elements of A and B.

Notation. Capital case is used for matrices and lower case Greek letters for

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scalars. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices, $\mathbb{C}^m=\mathbb{C}^{m\times 1}$ and $\mathbb{C}=\mathbb{C}^1$. A^T and A^H stand for transpose and conjugate transpose of A, respectively. $I^{(n)}$ is $n\times n$ identity matrix, and $0^{(n)}$ $n\times n$ null matrix. $A>0(\geqslant 0)$ denotes that A is a positive definite (positive semi-definite) Hermitian matrix. The column space of A is denoted by $R(A)$. $\|\ \|_2$ denotes the usual Euclidean vector norm and the spectral norm, and $\|\ \|_F$ the Frobenius matrix norm. $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the maximal singular value and the minimal singular value of A, respectively; and $\sigma^+_{\min}(A)$ is the minimal non-zero singular value of A. $G_{1,2}$ denotes the complex projective plane. The chordal distance between the points (α, β) and (γ, δ) on $G_{1,2}$ is

$$\rho((\alpha, \beta), (\gamma, \delta))=\frac{|\alpha\delta-\beta\gamma|}{\sqrt{(|\alpha|^2+|\beta|^2)(|\gamma|^2+|\delta|^2)}}.$$

§ 1. Generalized Singular Values and GSVD

We begin with the generalized eigenvalue concept.

Definition 1.1[9]. *Let $A, B\in\mathbb{C}^{m\times n}$, and*

$$\max_{(\lambda,\mu)\in G_{1,2}} \text{rank}\,(\mu A-\lambda B)=k.$$

A number-pair $(\alpha, \beta)\in G_{1,2}$ is an eigenvalue of the pencil $\mu A-\lambda B$ if rank $(\beta A-\alpha B)<k$.

The set of all eigenvalues of $\mu A-\lambda B$ is denoted by $\lambda(A, B)$.

Theorem 1.1. *Let $H, K\in\mathbb{C}^{n\times n}$, and $H, K\geqslant 0$. If*

$$\max_{\sigma,\tau\geqslant 0} \text{rank}(\tau H+\sigma K)=k, \tag{1.1}$$

then there exists a non-singular $S\in\mathbb{C}^{n\times n}$ such that

$$H=S\Lambda S^H,\ K=S\Omega S^H, \tag{1.2}$$

where

$$\Lambda=\text{diag}(\Lambda_1, 0),\ \Omega=\text{diag}(\Omega_1, 0), \tag{1.3}$$

$$\Lambda_1=\text{diag}(I^{(r)}, \Lambda_{10}, 0^{(k-r-s)}),\ \Omega_1=\text{diag}(0^{(r)}, \Omega_{10}, I^{(k-r-s)}), \tag{1.4}$$

$$\left.\begin{array}{c}\Lambda_{10}=\text{diag}(\alpha^2_{r+1}, \cdots, \alpha^2_{r+s}),\ \Omega_{10}=\text{diag}(\beta^2_{r+1}, \cdots, \beta^2_{r+s}),\\ 1>\alpha_{r+1}\geqslant\cdots\geqslant\alpha_{r+s}>0,\ 0<\beta_{r+1}\leqslant\cdots\leqslant\beta_{r+s}<1,\\ \alpha_i^2+\beta_i^2=1,\ r+1\leqslant i\leqslant r+s\end{array}\right\} \tag{1.5}$$

and $r, s\geqslant 0$, $r+s\leqslant k\leqslant n$.

Proof. From (1.1) there exist $\sigma, \tau\geqslant 0$ satisfying $\sigma^2+\tau^2=1$ such that $\text{rank}(\tau H+\sigma K)=k$. Let

$$\widetilde{H}=\sigma H-\tau K,\ \widetilde{K}=\tau H+\sigma K. \tag{1.6}$$

Then there is a non-singular $Q\in\mathbb{C}^{n\times n}$ such that

$$K_0=Q\widetilde{K}Q^H=\begin{pmatrix} I^{(k)} & 0\\ 0 & 0\end{pmatrix},\ H_0=Q\widetilde{H}Q^H=\begin{pmatrix} H_{11} & H_{12}\\ H_{12}^H & H_{22}\end{pmatrix}. \tag{1.7}$$

Suppose that $\eta_0 I+H_{11}>0$ for a certain $\eta_0>0$. Let

$$L=\begin{pmatrix} I & 0\\ -H_{12}^H(\eta I+H_{11})^{-1} & I\end{pmatrix},\ \eta\geqslant\eta_0.$$