A CLASS OF TWO-STAGE IMPLICIT HYBRID METHODS FOR ORDINARY EQUATIONS*

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Abstract

A k-step, $(k+2)$th order two-stage implicit hybrid method which has all the advantages of Enright's method but not its principal disadvantages is proposed. A "simple" approach to estimate the local truncation error is developed. Preliminary numerical results indicate that the hybrid method compares favorably with Enright's method.

1. Introduction

In this paper we shall propose a class of two-stage implicit hybrid multistep methods. The main reason is that they are able to replace the existing second derivative multistep methods which are suitable for the approximate numerical integration of stiff systems of first order ordinary differential equations and to overcome the main shortcoming of the latter. To show this, first of all we discuss a second derivative multistep method of the following type:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j y'_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j y''_{n+j}, \quad k=1,\ 2,\ \cdots \tag{1.1}$$

for the numerical integration of the stiff systems

$$y' = f(y), \quad t \in [0,\ T] \tag{1.2}$$

with the initial conditions

$$y(0) = y_0, \tag{1.3}$$

where the α_j, β_j and γ_j are constants normalized by making $\alpha_k = 1$, y_{n+j} is the approximate numerical solution obtained at t_{n+j}.

Note that y is a vector, although sometimes we consider only the scalar case. Suppose for the moment that $y(t)$ has a convergent Taylor series expansion at the point $t = t_n$. Consider the expansion

$$L[y(t_n);\ h] = \sum_{j=0}^{k} \alpha_j y(t_n + jh) - \sum_{j=0}^{k} \beta_j y'(t_n + jh) - h^2 \sum_{j=0}^{k} \gamma_j y''(t_n + jh)$$
$$= C_0 y(t_n) + C_1 h y'(t_n) + \cdots + C_p h^p y^{(p)}(t_n) + \cdots,$$

where $h = t_{j+1} - t_j$, $j = 0,\ 1,\ 2,\ \cdots$, is the step length and

$$\begin{cases} C_0 = \sum_{j=0}^{k} \alpha_j, \\ C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j), \end{cases} \tag{1.4}$$

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$$C_i=\sum_{j=0}^{k}\left(j^i\frac{\alpha_j}{i!}-j^{(i-1)}\frac{\beta_j}{(i-1)!}-j^{(i-2)}\frac{\gamma_j}{(i-2)!}\right),\ \text{for } i=2,3\cdots.$$

Definition 1. *If $C_0=C_1=\cdots=C_p=0$ but $C_{p+1}\neq 0$, then (1.1) is said to have order p. Thereafter the expression (1.4) is said to be the order condition of the method (1.1).*

If (1.1) is of order p, and we solve (1.1) for y_{n+k} with exact y_n, y_{n+1}, $\cdots$, y_{n+k-1}, i. e. with $y_{n+i}=y(t_{n+i})$ for $i=0(1)(k-1)$, where $y(t)$ is the solution of (1.2), then we have

$$y_{n+k}=y(t_{n+k})+O(h^{p+1}). \tag{1.5}$$

A count of the available coefficients shows that order $(3k+1)$ is attainable for the method (1.1) if it is implicit. However, the coefficients α_j, $j=0(1)k$, must satisfy the usual zero-stable condition, and this may prevent the above orders from being attained for some k. At present the maximum order for the zero-stable k-step method (1.1) $(k=1,\ 2,\ \cdots)$ is still unknown.

Some particular cases of (1.1) have been discussed. For example the fourth order method

$$y_{n+1}=y_n+\frac{1}{2}h(y'_n+y'_{n+1})+\frac{1}{12}h^2(y''_n-y''_{n+1}) \tag{1.6}$$

has been considered by Obrechkoff [2] and in connection with stiff systems by Ehle[3], Thompson [4] and others. This method is A-stable but not stable at infinity. Liniger and Willoughby [5] have considered the two-parameter method

$$y_{n+1}=y_n+\frac{h}{2}[(1-a)y'_n+(1+a)y'_{n+1}]+\frac{h^2}{4}[(b-a)y''_n-(b+a)y''_{n+1}]. \tag{1.7}$$

If a and b satisfy $0\leqslant a\leqslant 1/3$ and $b=1/3$ respectively, then this method is A-stable and is of order three at least. In particular, if $a=b=1/3$ then the obtained third order method is A-stable and stable at infinity.

Enright[6] attempted to derive for (1.1) stiffly stable methods satisfying the following three principal requirements:

(1) stability at infinity;

(2) a reasonable stability property in the neighborhood of the origin;

(3) an order as high as possible.

He obtained a k-step, $(k+2)$th order method of the following type:

$$y_{n+k}=y_{n+k-1}+h\sum_{j=0}^{k}\beta_j y'_{n+j}+h^2\gamma_k y''_{n+k},\quad k\leqslant 7. \tag{1.8}$$

Note that for $k=1$ the method corresponds to (1.7) with $a=b=1/3$.

The coefficients and plots of the stability regions for (1.8) for $k\leqslant 7$ are given in [6]. Thereafter (1.8) is called Enright's method.

The iteration scheme adopted to solve the implicit set of equations (1.8) is a modified Newton–Raphson technique:

$$W_{n+k}(y_{n+k}^{(l+1)}-y_{n+k}^{(l)})=-y_{n+k}^{(l)}+h\beta_k f(y_{n+k}^{(l)})+h^2\gamma_k\left(\frac{\partial f}{\partial y}\right)f(y_{n+k}^{(l)})$$
$$+y_{n+k-1}+h\sum_{j=0}^{k-1}\beta_j f_{n+j}, \tag{1.9}$$

where $W_{n+k}=\left(1-h\beta_k\left(\frac{\partial f}{\partial y}\right)-h^2\gamma_k\left(\frac{\partial f}{\partial y}\right)^2\right)$. In fact, for non-linear stiff systems the iteration schen e (1.9) neglects the terms involving $\left(\frac{\partial^2 f}{\partial y^2}\cdot f\right)$ in W_{n+k}.