A METHOD FOR CONSTRUCTING A CONTROLLABLE THIRD ORDER CURVE*

LI XIN-CAN (李心灿)

(Beijing Institute of Aeronautics and Astronautics, Beijing, China)

1. Problem Description

There exists, on the interval $[a, b]$, a partition

$$\Delta:\ a=x_1<x_2<\cdots<x_n=b$$

and a set of real numbers y_j, y'_j, $j=1, 2, \cdots, n$, such that its corresponding piecewise third order Hermite interpolating function is as follows:

$$H(x)=y_j f_0(x, j)+y_{j+1}f_1(x, j)+y'_j h_{j+1}g_0(x, j)+y'_{j+1}h_{j+1}g_1(x, j), \tag{1}$$

$$x_j\leqslant x\leqslant x_{j+1},\ h_{j+1}=x_{j+1}-x_j, \quad j=1, 2, \cdots, n-1$$

in which

$$f_0(x, j)=2\left(\frac{x-x_j}{h_{j+1}}\right)^3-3\left(\frac{x-x_j}{h_{j+1}}\right)^2+1, \quad f_1(x, j)=-2\left(\frac{x-x_j}{h_{j+1}}\right)^3+3\left(\frac{x-x_j}{h_{j+1}}\right)^2,$$

$$g_0(x, j)=\left(\frac{x-x_j}{h_{j+1}}\right)^3-2\left(\frac{x-x_j}{h_{j+1}}\right)^2+\left(\frac{x-x_j}{h_{j+1}}\right), \quad g_1(x, j)=\left(\frac{x-x_j}{h_{j+1}}\right)^3-\left(\frac{x-x_j}{h_{j+1}}\right)^2.$$

For many engineering problems, as the derivatives y'_j at data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$, are not given[2], formula (1) is not directly applicable.

To evaluate y'_j, a condition of C^2-continuity must be imposed at x_j, $j=2, 3, \cdots, n-1$. The resultant system of tridiagonal simultaneous equations ($n-2$ in number) plus two end conditions determine uniquely a set of $y'_j(j=1, 2, \cdots, n)$. This is the well-known third order spline function in terms of its first derivatives.

This kind of functions has extensive use in engineering, but some deficiencies are also felt. For instance, without preservation of convexity[3], unwanted points of inflexions would occur and the local properties would vanish. This will bring forth some inconvenience in design. The third order B-spline curve has some desirable properties[5, 6, 7], but it does not pass through data points (though we can make a curve passing through these points by finding the vertices first), and so the problem remains inconvenient. Consequently, it is a common desire to develop a sort of curves which preserves convexity and local properties, is easy to compute, and is suitable for interactive design.

This paper starts with formula (1) to define a third order curve and the control coefficients and then puts forward the necessary and sufficient conditions for the convexity of the curve. The curve is supposed to be C^1-continuous, manageable in getting straight line segments, cuspidal points and inflexion points, and can be expressed explicitly in terms of coordinates of data points (and end conditions).

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2. Representations of the Curve

Let there be a set of data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$, with two given end conditions. It is required to construct a curve satisfying these conditions. As the first derivatives are not given at these data points, formula (1) is not directly applicable.

To provide additional constraints and to facilitate the adjustment and control of the shape of the curve, we take

$$y'_j=(1-\lambda)\frac{y_{j+1}-y_j}{h_{j+1}}+\lambda\frac{y_j-y_{j-1}}{h_j},\quad y'_{j+1}=(1-\mu)\frac{y_{j+2}-y_{j+1}}{h_{j+2}}+\mu\frac{y_{j+1}-y_j}{h_{j+1}},\tag{2}$$

$$j=1, 2, \cdots, n-1 \text{ and } 0\leqslant\lambda\leqslant1,\ 0\leqslant\mu\leqslant1,$$

where λ and μ are control coefficients. We will see later that by the adjustment of these two coefficients, the shape of a curve can be manipulated at will and unwanted points of inflexions can be eliminated.

In formula (2), owing to the presence of the four undetermined quantities y_0, y_{n+1}, h_1, h_{n+1}, we make extensions on both ends: $P_0(x_0, y_0)$, $x_0=x_1-h_1$ at the left end and $P_{n+1}(x_{n+1}, y_{n+1})$, $x_{n+1}=x_n+h_{n+1}$ at the right end, with $h_1=h_2$, $h_{n+1}=h_n$. As for y_0, y_{n+1}, they are determined as follows:

(I) If the end conditions are y'_1, y'_n, then y_0, y_{n+1} should satisfy

$$y'_1=(1-\lambda)\frac{y_2-y_1}{h_2}+\lambda\frac{y_1-y_0}{h_1},\quad y'_n=(1-\mu)\frac{y_{n+1}-y_n}{h_{n+1}}+\mu\frac{y_n-y_{n-1}}{h_n}.\tag{3}$$

(II) If the end conditions are y''_1, y''_n, then y_0, y_{n+1} should satisfy

$$y''_1=[4\lambda\quad -2(1-\mu)]\begin{bmatrix}\dfrac{\dfrac{y_2-y_1}{h_2}-\dfrac{y_1-y_0}{h_1}}{h_2} & \dfrac{\dfrac{y_3-y_2}{h_3}-\dfrac{y_2-y_1}{h_2}}{h_2}\end{bmatrix}^T,$$

$$y''_n=[-2\lambda\quad 4(1-\mu)]\begin{bmatrix}\dfrac{\dfrac{y_n-y_{n-1}}{h_n}-\dfrac{y_{n-1}-y_{n-2}}{h_{n-1}}}{h_n}\\[2ex] \dfrac{\dfrac{y_{n+1}-y_n}{h_{n+1}}-\dfrac{y_n-y_{n-1}}{h_n}}{h_n}\end{bmatrix}.\tag{4}$$

(III) If the end conditions are not specified, y_0, y_{n+1} should satisfy

$$\frac{y_2-y_1}{h_2}-\frac{y_1-y_0}{h_1}=\left(\frac{y_3-y_2}{h_3}-\frac{y_2-y_1}{h_2}\right)^2\bigg/\left(\frac{y_4-y_3}{h_4}-\frac{y_3-y_2}{h_3}\right),$$

$$\frac{y_{n+1}-y_n}{h_{n+1}}-\frac{y_n-y_{n-1}}{h_n}=\left(\frac{y_n-y_{n-1}}{h_n}-\frac{y_{n-1}-y_{n-2}}{h_{n-1}}\right)^2\bigg/\left(\frac{y_{n-1}-y_{n-2}}{h_{n-1}}-\frac{y_{n-2}-y_{n-3}}{h_{n-2}}\right).\tag{5}$$

We can thus express piecewisely, in explicit form, the interpolating functions passing through the data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$.