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FINITE DIFFERENCE METHOD OF THE BOUNDARY PROBLEM FOR THE SYSTEMS OF SOBOLEV-GALPERN TYPE*

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§1

Many authors have paid great attention to the study of the linear and nonlinear pseudo-parabolic equations or the equations of Sobolev-Galpern type. The nonlinear pseudo-parabolic equations of ten occur in practical research, such as the equations for the long waves in nonlinear dispersion systems, the equations in the cooling process according to two-temperature of heat conduction, the equations for filtration of fluids in the broken rock and so forth[1-10]. These equations contain the differential operator $u_t - u_{xxt}$ as the main part. Some fairly general family of nonlinear pseudo-parabolic systems[11,12], which contain the above mentioned equations as special cases, are considered by Galerkin's method.

Now let us consider in rectangular domain $Q_T=\{0\leqslant x\leqslant l, 0\leqslant t\leqslant T\}$ the nonlinear pseudo-parabolic system or the system of Sobolev-Galpern type

$$(-1)^M u_t + A(x, t) u_{x^{2M}t} = B(x, t, u, \cdots, u_{x^{2M-1}}) u_{x^{2M}} + F(x, t, u, \cdots, u_{x^{2M-1}}) \tag{1}$$

with the boundary condition

$$u_{x^k}(0, t) = u_{x^k}(l, t) = 0, \quad k=0, 1, \cdots, M-1 \tag{2}$$

and the initial condition

$$u(x, 0) = \varphi(x), \tag{3}$$

where $u(x, t) = (u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector valued unknown function.

Suppose that the following assumptions are fulfilled.

(I) $A(x, t)$ is a $m\times m$ symmetric positively definite continuous matrix and has bounded derivative $A_t(x, t)$; i.e., for any m-dimensional vectors $\xi\in\mathbb{R}^m$, $(\xi, A\xi) \geqslant a|\xi|^2$, where $a>0$.

(II) $B(x, t, p_0, p_1, \cdots, p_{2M-1})$ is a $m\times m$ semibounded continuous matrix of variables $(x, t)\in Q_T$ and $p_0, p_1, \cdots, p_{2M-1}\in\mathbb{R}^m$, i.e., there exists a constant b, such that

$$(\xi, B(x, t, p_0, p_1, \cdots, p_{2M-1})\xi) \leqslant b|\xi|^2 \tag{4}$$

for any m-dimensional vectors $\xi\in\mathbb{R}^m$ and $(x, t)\in Q_T$, $p_0, p_1, \cdots, p_{2M-1}\in\mathbb{R}^m$.

(III) $F(x, t, p_0, p_1, \cdots, p_{2M-1})$ is a m-dimensional vector valued continuous function satisfying the relation

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$$|F(x,\ t,\ p_0,\ p_1,\ \cdots,\ p_{2M-1})|\leqslant K_1\left\{\sum_{k=0}^{2M-1}|p_k|+1\right\},\tag{5}$$

where K_1 is a constant.

(IV) $\varphi(x)$ is a m-dimensional vector valued initial function, belonging to $C^{(2M)}([0,\ l])$ and satisfying the homogeneous boundary condition (2).

In [13] it is proved by the fixed point technique, that under the conditions (I), (II), (III) and $\varphi(x)\in W_2^{(2M)}(0,\ l)$, the boundary problem (2), (3) for the nonlinear pseudo-parabolic system (1) has at least one m-dimensional vector valued global solution $u(x,\ t)$ in the functional space $W_\infty^{(1)}((0,\ T);\ W_2^{(2M)}(0,\ l))$. In addition that $B(x,\ t,\ p_0,\ p_1,\ \cdots,\ p_{2M-1})$ and $F(x,\ t,\ p_0,\ p_1,\ \cdots,\ p_{2M-1})$ are continuously differentiable or more precisely are locally Lipschitz continuous with respect to $p_0,\ p_1,\ \cdots,\ p_{2M-1}\in\mathbb{R}^m$, the solution $u(x,\ t)$ is unique.

The purpose of this note is to study the boundary problem (2), (3) for the system (1) by the finite difference method under the above mentioned assumptions (I), (II), (III) and (IV).

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The finite interval $[0,\ l]$ can be divided into the small segment grids by the points $x_j=jh(j=0,\ 1,\ \cdots,\ J)$, where $Jh=l$, J is an integer and h is the stepsize. The discrete function $\{u_j\}$ $(j=0,\ 1,\ \cdots,\ J)$ is defined on the grid points $x_j(j=0,1,\cdots,J)$. We denote the scalar product of two discrete functions $\{u_j\}$ and $\{v_j\}$ by $(u,v)_h=\sum_{j=0}^{J}u_jv_jh$. And $\|u\|_h=(u,\ u)_h$. Also we introduce the symbol $\|u\|_\infty=\max_{j=0,1,\cdots,J}|u_j|$.

Let us denote $\Delta_+u_j=u_{j+1}-u_j$ and $\Delta_-u_j=u_j-u_{j-1}(j=0,\ 1,\ \cdots,\ J)$. Similarly we take

$$\left\|\frac{\Delta_+u}{h}\right\|_h^2=\sum_{j=0}^{J-1}\left|\frac{\Delta_+u_j}{h}\right|^2h=\sum_{j=1}^{J}\left|\frac{\Delta_-u_j}{h}\right|^2h=\left\|\frac{\Delta_-u}{h}\right\|_h^2$$

and they can be denoted simply by $\|\delta_hu\|_h$. Also we have $\left\|\frac{\Delta_+u}{h}\right\|_\infty=\left\|\frac{\Delta_-u}{h}\right\|_\infty=\|\delta_hu\|_\infty$. We adopt the similar notations for the difference quotients of higher order: $\|\delta_h^ku\|_h$ and $\|\delta_h^ku\|_\infty$ $(k=0,\ 1,\ \cdots)$.

Now we state some lemmas which are useful for later discussions and whose proof can be found in [14].

Lemma 1. *For any two discrete functions $\{u_j\}$ and $\{v_j\}$ $(j=0,\ 1,\ \cdots,\ J)$ on finite interval, there are identities*

$$\sum_{j=0}^{J-1}u_j\Delta_+v_j=-\sum_{j=1}^{J}v_j\Delta_-u_j-u_0v_0+u_Jv_J,\tag{6}$$

$$\sum_{j=1}^{J-1}u_j\Delta_+\Delta_-v_j=-\sum_{j=0}^{J-1}(\Delta_+u_j)(\Delta_+v_j)-u_0\Delta_+v_0+u_J\Delta_-v_J.\tag{7}$$

Lemma 2. *For any discrete function $\{u_j\}$ $(j=0,\ 1,\ \cdots,\ J)$ on the finite interval $[0,l]$, there are the interpolation relations*

$$\|\delta_h^ku\|_h\leqslant K_2\|u\|_h^{1-\frac{k}{n}}\left(\|\delta_h^nu\|_h+\frac{\|u\|_h}{l^n}\right)^{\frac{k}{n}},\quad k=0,\ 1,\ \cdots,\ n\tag{8}$$