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ROBUST HIGH ORDER CONVERGENCE OF AN OVERLAPPING SCHWARZ METHOD FOR SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEMS*

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Abstract

In this article we propose an overlapping Schwarz domain decomposition method for solving a singularly perturbed semilinear reaction-diffusion problem. The solution to this problem exhibits boundary layers of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$ at both ends of the domain due to the presence of singular perturbation parameter ε. The method splits the domain into three overlapping subdomains, and uses the Numerov or Hermite scheme with a uniform mesh on two boundary layer subdomains and a hybrid scheme with a uniform mesh on the interior subdomain. The numerical approximations obtained from this method are proved to be almost fourth order uniformly convergent (in the maximum norm) with respect to the singular perturbation parameter. Furthermore, it is proved that, for small ε, one iteration is sufficient to achieve almost fourth order uniform convergence. Numerical experiments are given to illustrate the theoretical order of convergence established for the method.

Mathematics subject classification: 65L10, 65L11, 65L20.
Key words: Singular perturbation, Semilinear reaction-diffusion, Overlapping Schwarz method, Robust convergence, Numerov scheme.

1. Introduction

We consider singularly perturbed semilinear reaction-diffusion problem of the form

$$Tu := -\varepsilon u'' + f(x,u) = 0, \quad x \in \Omega = (0,1), \tag{1.1a}$$

$$u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{1.1b}$$

where $\varepsilon \ll 1$ is a small positive parameter and f is a sufficiently smooth function. In general, as ε tends to zero, the solution u of (1.1) may exhibit boundary and/or internal layers of various types. The location of the layers and the behavior of the solution in these layers depend on the character of f [5,13]. Problems of type (1.1) are probably the most frequently studied singular perturbation problems, both asymptotically and numerically, see [5,9,13,15,16,19,21–23], and the references therein. This interest can be justified by several model problems arising in many areas of science and engineering, such as theory of nonpremixed combustion [24], Michaelis-Menten process in biology [4], and catalytic reaction theory [1].

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We consider problem (1.1) with the assumption that

$$f_u(x,y) \geq \beta > 0 \quad \text{for all } (x,y) \in \Omega \times \mathbb{R}. \tag{1.2}$$

Under this assumption, problem (1.1) and the reduced problem $f(x,u_0(x)) = 0,\ x \in \Omega$, have unique solutions u and u_0 respectively. The solution u generally has exponential boundary layers at $x = 0$ and $x = 1$ of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$. More precisely, u can be decomposed into two parts: $u = v + w$, where for $s = 0,\ldots,6$ and $x \in \overline{\Omega}$ [20]

$$|v^{(s)}(x)| \leq C, \qquad |w^{(s)}(x)| \leq C\varepsilon^{-s/2}\left(e^{(-x\sqrt{\beta/\varepsilon})} + e^{(-(1-x)\sqrt{\beta/\varepsilon})}\right). \tag{1.3}$$

Due to the presence of thin layers, classical numerical methods fail to produce satisfactory results for sigularly perturbed problems, when the perturbation parameter is sufficiently small. This leads to the development of special numerical methods – so called 'parameter-robust' or 'uniformly convergent' numerical methods – that behave uniformly well for all values of the perturbation parameter, no matter how small [7, 17]. Farrell et al. [8] gave a theoretical result which shows that, even in the case of very simple nonlinearity of $f(x,u)$ in u, that is, $f(x,u) = a(u)u$, uniform convergence cannot be achieved in the discrete maximum norm using fitted finite difference schemes on uniform meshes. Most of the works for singularly perturbed semilinear reaction-diffusion problems involve the use of standard finite difference method on special layer-resolving meshes [17]. In the current work we consider a Schwarz domain decomposition method with overlapping subdomains. The origins of this approach can be traced back to the seminal work of Schwarz [18] in the nineteenth century. However, the general development of domain decomposition algorithms occurred only subsequent to the development of parallel computer architectures. More details on domain decomposition can be found in [14].

Suitably designed Schwarz domain decomposition methods have been proven to yield uniformly accurate results for singularly perturbed semilinear reaction-diffusion problems [2, 3, 6]. These methods involve the use of either overlapping subdomains or two overlapping sets of subdomains with no subdomain overlap within each set. The order of uniform convergence in these works is not higher than two. In the current work we design an overlapping Schwarz domain decomposition method that yields almost fourth order uniform approximations for the solution of (1.1). A comprehensive analysis has been given to prove the uniform convergence (in the maximum norm) with respect to the perturbation parameter. Furthermore, we address much faster convergence of the iterative process for small ε. More specifically, it is shown that, for small ε, one iteration is sufficient to achieve almost fourth order uniform convergence.

This paper is organized as follows. In Section 2, we propose an overlapping Schwarz domain decomposition method for solving problem (1.1), and establish nodal error estimates and global error estimates. Results of the numerical experiments are given in Section 3.

Notation. Throughout the paper, C denotes a generic positive constant that is independent of ε, k and the discretization parameter N. We consider the maximum norm and denote it by $||.||_D$, where D is a closed and bounded subset of $\overline{\Omega}$. For a real valued function $g \in C(D)$, we define $||g||_D = \max_{x\in D}|g(x)|$. If $D = \overline{\Omega}$, we drop D from the notation. The analogous discrete maximum norm on the mesh $\overline{\Omega}^N$ is denoted by $||.||_{\overline{\Omega}^N}$. For any functions $g, y_p \in C(\overline{\Omega})$, define $g_i = g(x_i)$, $y_{p,i} = y_p(x_i)$.