Journal of Computational Mathematics
Vol.35, No.2, 2017, 169–186.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1608-m2015-0279

# STRUCTURED CONDITION NUMBERS FOR THE TIKHONOV REGULARIZATION OF DISCRETE ILL-POSED PROBLEMS*

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**Abstract**

The possibly most popular regularization method for solving the least squares problem $\min_x \|Ax-b\|_2$ with a highly ill-conditioned or rank deficient coefficient matrix $A$ is the Tikhonov regularization method. In this paper we present the explicit expressions of the normwise, mixed and componentwise condition numbers for the Tikhonov regularization when $A$ has linear structures. The structured condition numbers in the special cases of nonlinear structure i.e. Vandermonde and Cauchy matrices are also considered. Some comparisons between structured condition numbers and unstructured condition numbers are made by numerical experiments. In addition, we also derive the normwise, mixed and componentwise condition numbers for the Tikhonov regularization when the coefficient matrix, regularization matrix and right-hand side vector are all perturbed, which generalize the results obtained by Chu et al. [Numer. Linear Algebra Appl., 18 (2011), 87–103].

*Mathematics subject classification:* 65F35, 65F20.
*Key words:* Tikhonov regularization, Discrete ill-posed problem, Structured least squares problem, Structured condition number.

## 1. Introduction

Consider the problem of determining an approximate solution of the least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|_2 \tag{1.1}$$

with a right-hand side vector $b\in\mathbb{R}^m$, which has been contaminated by an error $e\in\mathbb{R}^m$, and a matrix $A\in\mathbb{R}^{m\times n}$ of ill-determined rank, i.e., $A$ has many singular values of different size close to the origin. In particular, $A$ is severely ill-conditioned and may be rank deficient. We allow $m\geq n$, and do not require $b$ to be in the range of $A$. Minimization problems with a matrix of ill-determined rank, such as (1.1), are often referred to as discrete ill-posed problems. They arise from the discretization of linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel; see, e.g., Engl et al. [9], Hansen [12], and Morozov [17] for discussions on ill-posed problems, their discretization, and approximate solution.

Let $\hat{b}\in\mathbb{R}^m$ be the (unknown) error-free vector associated with $b$, i.e., $b=\hat{b}+e$. We are interested in computing an approximation of $\tilde{x}=A^\dagger\hat{b}$, the minimal-norm solution of the error-free least squares problem associated with (1.1), where $A^\dagger$ denotes the Moore-Penrose inverse of $A$. The minimal-norm least squares solution $A^\dagger b$ of (1.1) is not a meaningful approximation of $\tilde{x}$ due to the error in $b$ and the ill-conditioning of $A$. This difficulty is remedied by replacing

* Received May 5, 2015 / Revised version received May 2, 2016 / Accepted August 19, 2016 / Published online March 10, 2017 /
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the problem (1.1) by a nearby system that is less sensitive to the error $e$. This replacement is referred to as regularization. Perhaps the best known regularization method is the Tikhonov regularization, which replaces system (1.1) by a penalized least-squares problem of the form

$$\min_x \left\{ \|Ax-b\|_2^2 + \lambda^2 \|Lx\|_2^2 \right\}, \tag{1.2}$$

with regularization matrix $L \in \mathbb{R}^{p\times n}$ and regularization parameter $\lambda > 0$. The easiest regularization matrix is $L = I_n$ ($I_n$ denotes the $n\times n$ identity matrix), which is known as the standard form. The solution to (1.2) is called the Tikhonov regularization solution to (1.1).

Given a problem, the condition number measures the worst-case sensitivity of its solution to small perturbations in the input data. Using it with backward error estimate, we can derive an approximate upper bound for the forward error, i.e., the difference between a perturbed solution and the exact solution, thanks to the well-known "rule of thumb" (cf. [14]),

$$\text{forward error} \lesssim \text{condition number} \times \text{backward error}.$$

For the Tikhonov regularization, Malyshev [16] studied the normwise condition number for the standard form (i.e., $L = I_n$) when only the coefficient matrix $A$ or the right-hand side $b$ is perturbed. Recently, Chu et al. [6] derived the normwise condition number for the Tikhonov regularization when both $A$ and $b$ are perturbed.

However, the normwise condition number ignores the features of both input and output data with respect to scaling and/or sparsity. When the data is badly scaled or contains many zeros, measuring the size of a perturbation in terms of its norm leaves us in the dark concerning the relative size of the perturbation on its small (or zero) entries. To tackle this drawback, two kinds of condition numbers are introduced: mixed and componentwise condition numbers. The former measures the errors in output using norms and the input perturbations componentwise, and the latter measures both the errors in output and the perturbations in input componentwise. Hence, the mixed and componentwise condition numbers of Tikhonov regularization are also taken into consideration in [6] when both $A$ and $b$ are perturbed.

In many applications, the matrix $A$ in problem (1.1) has a special structure. The structured matrices such as Toeplitz and Cauchy matrices arise in a variety of applications, for instance, numerical differential equations, stationary autoregressive time series, numerical integral equations of convolution-type, system identification problems, and image restoration problems [3–5, 18], to mention just a few. When $A$ is structured, it is reasonable to require the perturbation $\Delta A$ of $A$ has the same structure. However, the requirement of special structures was not considered in the condition numbers for the Tikhonov regularization studied in [6, 16] and hence, it's unreasonable to apply the unstructured condition number to the Tikhonov regularization of the structured least squares problem. Motivated by this, it is necessary to study the condition numbers for the Tikhonov regularization of the structured least squares problem. To the best of our knowledge, there is no structured condition numbers available for the Tikhonov regularization of the structured least squares problem. Our paper is to fill in this gap. We will derive the normwise, mixed and componentwise condition numbers for the Tikhonov regularization of the structured least squares problem. Note that, the authors in [6] derived the normwise, mixed and componentwise condition numbers when both $A$ and $b$ are perturbed while the regularization matrix $L$ does not perturbed. Therefore, the normwise, mixed and componentwise condition numbers for the Tikhonov regularization in the case that $A$, $L$ and $b$ are all perturbed are also discussed in this paper, which generalize the results obtained by Chu